Solving exponential equations

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Summary

This guide applies the laws of indices to solve equations involving powers; a key skill in mathematics and many areas of science.

It is highly recommended that you read Guide: Laws of indices before reading this guide. In addition, Guide: Logarithms is recommended but not required for some of the material in this guide.

In Guide: Laws of indices, you learned about the different ways that you can manipulate expressions involving powers (like x^3 and 2^{80}) and nth roots (such as $\sqrt[4]{x}$ and $2^{1/80}$). In general mathematical life however, this knowledge will not be enough; you will need to apply these laws of indices to help you solve equations.

Solving equations involving indices is a key skill in areas whenever exponential growth or decay plays a role: economics, physics (particularly nuclear physics), chemistry, biology, and many more.

Before you get started however, it is worth reiterating the recommendation above:

> **!** Important
>
> **This guide assumes an excellent knowledge of the laws of indices. Please make sure that you have read Guide: Laws of indices before continuing.**
> The numbering of the laws will follow the numbering in this guide and on Factsheet: Laws of indices.
> In addition, an initial understanding of logarithms will be required later in the guide. While this understanding is partially explained in this guide, you may want to familiarize yourself with Guide: Logarithms before attempting Examples 7, 8, 9, and 10.

Discussion of techniques

In general, the golden rule of solving equations involving indices is the following:

Tip

Make sure both sides are the in the same base before simplifying. This is because if a is a positive number not equal to 1 and $a^m = a^n$, then you can immediately say that $m = n$ and solve the equation from there.

However, you may be wondering; why on earth is this true? How can you ignore the base in this case? Or, more pertinently, *what if you can't write the equation in a form where both sides have the same base*?

What you can do in this case is isolate the variable on one side of the equation, and have a constant on the other (see Guide: Rearranging equations; perhaps it looks like $a^x = b$. From there, you can take **logarithms** to base a of both sides.

Since for all $a > 0$ with $a \neq 1$ and all real numbers y:

$$a^{\log_a(y)} = y \qquad \text{and} \qquad \log_a(a^y) = y$$

it follows that

$$\log_a(a^x) = \log_a b$$

implies that

$$x = \log_a b$$

solving the equation! What this means is that logarithms undo exponentiation and exponentiation undoes logarithms.

In particular, this is why $a^m = a^n$ implies that $m = n$. Taking logarithms to base a of both sides of $a^m = a^n$ gives $\log_a a^m = \log_a a^n$; using the above result then yields $m = n$.

However, please be aware of the following:

Warning

You cannot take logarithms of a negative number! So an expression like $\log_a(-4)$ is **not** defined.

This is particularly pertinent in Example 9 below.

In this guide, logarithms are only used to undo exponentiation to solve equations; there will not be any applications of the laws of logarithms nor the change of base rule. (For more on these, see Guide: Logarithms.)

Examples

Initial examples

Example 1

Solve $x^{\frac{1}{2}} = 8$.

You can start squaring both sides of the equation to get $(x^{\frac{1}{2}})^2 = 8^2$. Using Law 3, $(x^{1/2})^2 = x^{2/2} = x$; so then you get the answer $x = 64$. To check if the answer is correct, you can substitute 64 back into the equation: $64^{\frac{1}{2}} = \sqrt[2]{64} = 8$.

Example 2

Solve $x^{-2} = 9$.

Using Law 5, $x^{-2} = 9$ can be re-expressed as $\frac{1}{x^2} = 9$. Multiplying both sides by x^2 gives you $1 = 9x^2$. Then dividing both sides by 9 gives you $\frac{1}{9} = x^2$. Remembering you can have positive and negative roots, you get $x = \frac{1}{3}$ or $x = -\frac{1}{3}$ as the two solutions to this equation.

Example 3

Solve $3^{4x} = 27^{5-x}$.

You can notice here that the two sides of the equation have different bases; so you need to write these in the same base. As $27 = 3^3$, the equation can be rewritten as: $3^{4x} = (3^3)^{5-x}$. Then using Law 3, you can write $3^{4x} = 3^{15-3x}$. Since the bases of both sides are equal, you can say that $4x = 15 - 3x$. Rearranging gives $x = 15/7$.

Example 4

Solve $x^{\frac{5}{3}} + 3x^{\frac{2}{3}} = 10x^{-\frac{1}{3}}$.

If you look at all of the indices in the question, the denominator is 3 and that is a hint of what you need to multiply by. Multiply by $x^{\frac{1}{3}}$ on both sides of the equation; using Law 1, this gives

$$x^{\frac{5}{3}} \cdot x^{\frac{1}{3}} + 3x^{\frac{2}{3}} \cdot x^{\frac{1}{3}} = 10x^{-\frac{1}{3}} \cdot x^{\frac{1}{3}}$$
$$x^{\frac{6}{3}} + 3x^{\frac{3}{3}} = 10x^{0}$$

Using Law 4, you can further simplify this expression to get $x^2 + 3x = 10$, and so $x^2 + 3x - 10 = 0$. This is a quadratic equation; factorizing this equation gives you $(x+5)(x-2) = 0$, then you can get $x = -5$ and $x = 2$ as the two possible solutions to this equation.

Example 5

Solve $2^{x+1} \cdot 3^x = 72$.

Here, you will need to condense the left-hand side into a single base. Using Law 1, you can write that

$$2^{x+1} \cdot 3^x = 2(2^x \cdot 3^x) = 72.$$

Next, use Law 7 to combine 2^x and 3^x into a single base;

$$2(2^x \cdot 3^x) = 2((2 \cdot 3)^x) = 2 \cdot 6^x = 72.$$

Therefore, $6^x = 36$. Since $36 = 6^2$, it follows that $x = 2$. (Or indeed, you could have taken logarithms of both sides to base 6.)

Example 6

Solve $\sqrt{(6x)^3} = 8\sqrt{27}$.
First of all, you can use Law 7 to write $(6x)^3 = 6^3 \cdot x^3$. Using Law 10, you can then write the left hand side of this equation as

$$\sqrt{(6x)^3} = \sqrt{6^3 \cdot x^3} = \sqrt{6^3} \cdot \sqrt{x^3}$$

You can now work on $\sqrt{6^3}$; using Law 7 and Law 10, you can write

$$\sqrt{6^3} = \sqrt{2^3 \cdot 3^3} = \sqrt{8} \cdot \sqrt{27}.$$

Therefore, the initial equation becomes

$$\sqrt{(6x)^3} = \sqrt{8} \cdot \sqrt{27} \cdot \sqrt{x^3} = 8\sqrt{27}.$$

Dividing both sides by $\sqrt{8}$ (thereby using Laws 2 and 6) and cancelling the $\sqrt{27}$ gives

$$\sqrt{x^3} = \frac{8}{\sqrt{8}} = \frac{8^1}{8^{1/2}} = 8^{1/2} = \sqrt{8}.$$

Squaring both sides to undo the square root gives $x^3 = 8$, and so $x = 2$.

Examples involving logarithms

Example 7

Solve $e^{3x} = 12$.
Taking the logarithm of both sides to base e gives you $\log_e(e^{3x}) = \log_e(12)$. Using the definition of logarithms, you can express the equation as $3x = \ln(12)$. Rearranging the equation gives you $x = \dfrac{\ln(12)}{3}$ and this is your final answer.

Tip

Although e here was treated as some constant, it is actually a very important constant called *Euler's number*.
In addition, there is a special name for $\log_e(x)$; this is the **natural logarithm** of x, often written $\ln(x)$.
To see more about Euler's number e and natural logarithms, please read Guide: Logarithms.

Example 8

Given the equation $6^x = 3^{x+1}$, solve for x.

First of all, you can notice that as 6 is not a power of 3 (or vice versa) getting these in the same base is difficult. What you can do instead is use the laws of indices to get an expression of the form $a^x = b$ and then take logarithms. Using Law 1, you can write that

$$6^x = 3^x \cdot 3^1 = 3(3^x)$$

Dividing both sides by 3^x and using Law 8 gives

$$\left(\frac{6^x}{3^x}\right) = \left(\frac{6}{3}\right)^x = 3$$

and so you are left with $2^x = 3$. Taking logarithms of both sides to base 2 gives $x = \log_2(3)$; and this is your final answer!

Example 9

Given the equation $5^{2x} + 7(5^x) - 30 = 0$, you are asked to solve for x.

Start by letting $y = 5^x$. Then, you can rewrite the equation given as $(5^x)^2 + 7(5^x) - 30 = 0$ using Law 3, which is the same as writing

$$y^2 + 7y - 30 = 0.$$

Recognizing that this is a quadratic equation (see Guide: Introduction to quadratic equations for more), you can use the quadratic formula or otherwise to show that

$$y = -10 \quad \text{or} \quad y = 3.$$

As $y = 5^x$, this means that $5^x = -10$ or $5^x = 3$. By taking the logarithm of both sides to base 5, you can show that

$$x = \log_5(-10) \quad \text{or} \quad x = \log_5(3).$$

Remember from above that the logarithm of a negative number is not defined, so $x = \log_5(3)$ is the only viable solution.

The final example uses many of the laws of indices!

Example 10

Solve $3^{3x} = 5^{x-4}$.

This example is similar to Example 8, only with a few extra steps. Once again, as 3 is not a power of 5 or vice versa, the strategy is to write the equation in the form $a^x = b$ and then take logarithms of both sides.

First of all, use Law 1 to write $5^{x-4} = 5^x \cdot 5^{-4}$ and Law 3 to write $3^{3x} = (3^3)^x$. Since $3^3 = 27$, the equation becomes

$$27^x = 5^x \cdot 5^{-4}$$

Dividing both sides by 5^x gives

$$\frac{27^x}{5^x} = 5^{-4}$$

Using Law 8 and Law 5, you can write

$$\left(\frac{27}{5}\right)^x = \frac{1}{5^4} = \frac{1}{625}$$

Taking logarithms to base $27/5$ on both sides gives $x = \log_{27/5}(1/625)$, which is your final answer.

Quick check problems

1. Solve $3x^3 \cdot 5 = 405$ for x.
2. Solve $(9x - 1)^{1/3} = 4$ for x.
3. Solve $x^{3/2} = \dfrac{125}{27}$ for x.

Further reading

For more questions on the subject, please go to Questions: Solving exponential equations.

Version history and licensing

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- v1.1: edited 04/24 by tdhc.