Solving exponential equations

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Summary

This guide applies the laws of indices to solve equations involving powers; a key skill in mathematics and many areas of science.

*It is highly recommended that you read* [*Guide: Laws of indices*](lawsofindices.qmd) *before reading this guide. In addition,* [*Guide: Logarithms*](logarithms.qmd) *is recommended but not required for some of the material in this guide.*

In [Guide: Laws of indices](lawsofindices.qmd), you learned about the different ways that you can manipulate expressions involving powers (like $x^{3}$ and $2^{80}$) and $n$th roots (such as $\sqrt[4]{x}$ and $2^{1/80}$). In general mathematical life however, this knowledge will not be enough; you will need to apply these laws of indices to help you solve equations.

Solving equations involving indices is a key skill in areas whenever exponential growth or decay plays a role: economics, physics (particularly nuclear physics), chemistry, biology, and many more.

Before you get started however, it is worth reiterating the recommendation above:

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|  Important |
| **This guide assumes an excellent knowledge of the laws of indices. Please make sure that you have read** [**Guide: Laws of indices**](lawsofindices.qmd) **before continuing.**The numbering of the laws will follow the numbering in this guide and on [Factsheet: Laws of indices](../factsheets/f-lawsofindices.qmd).In addition, an initial understanding of logarithms will be required later in the guide. While this understanding is partially explained in this guide, you may want to familiarize yourself with [Guide: Logarithms](logarithms.qmd) before attempting Examples 7, 8, 9, and 10. |

# Discussion of techniques

In general, the golden rule of solving equations involving indices is the following:

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|  Tip |
| **Make sure both sides are the in the same base before simplifying**. This is because if $a$ is a positive number not equal to $1$ and $a^{m}=a^{n}$, then you can immediately say that $m=n$ and solve the equation from there. |

However, you may be wondering; why on earth is this true? How can you ignore the base in this case? Or, more pertinently, *what if you can’t write the equation in a form where both sides have the same base*?

What you can do in this case is isolate the variable on one side of the equation, and have a constant on the other (see [Guide: Rearranging equations](introtorearrange%20for%20more); perhaps it looks like $a^{x}=b$. From there, you can take **logarithms** to base $a$ of both sides.

Since for all $a>0$ with $a\ne 1$ and all real numbers $y$:

$$ a^{log\_{a}\left(y\right)}=y   and   log\_{a}\left(a^{y}\right)=y$$

it follows that

$$log\_{a}\left(a^{x}\right)=log\_{a}b$$

implies that

$$x=log\_{a}b$$

solving the equation! What this means is that logarithms undo exponentiation and exponentiation undoes logarithms.

In particular, this is why $a^{m}=a^{n}$ implies that $m=n$. Taking logarithms to base $a$ of both sides of $a^{m}=a^{n}$ gives $log\_{a}a^{m}=log\_{a}a^{n}$; using the above result then yields $m=n$.

However, please be aware of the following:

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|  Warning |
| **You cannot take logarithms of a negative number!** So an expression like $log\_{a}\left(−4\right)$ is **not** defined. |

This is particularly pertinent in Example 9 below.

In this guide, logarithms are only used to undo exponentiation to solve equations; there will not be any applications of the laws of logarithms nor the change of base rule. (For more on these, see [Guide: Logarithms](logarithms.qmd).)

# Examples

## Initial examples

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|  | **Example 1**Solve $x^{\frac{1}{2}}=8$.You can start squaring both sides of the equation to get $\left(x^{\frac{1}{2}}\right)^{2}=8^{2}$. Using Law 3, $\left(x^{1/2}\right)^{2}=x^{2/2}=x$; so then you get the answer $x=64$. To check if the answer is correct, you can substitute $64$ back into the equation: $64^{\frac{1}{2}}=\sqrt[2]{64}=8$. |

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|  | **Example 2**Solve $x^{−2}=9$.Using Law 5, $x^{−2}=9$ can be re-expressed as $\frac{1}{x^{2}}=9$. Multiplying both sides by $x^{2}$ gives you $1=9x^{2}$. Then dividing both sides by $9$ gives you $\frac{1}{9}=x^{2}$. Remembering you can have positive and negative roots, you get $x=\frac{1}{3}$ or $x=−\frac{1}{3}$ as the two solutions to this equation. |

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|  | **Example 3**Solve $3^{4x}=27^{5−x}$.You can notice here that the two sides of the equation have different bases; so you need to write these in the same base. As $27=3^{3}$, the equation can be rewritten as: $3^{4x}=\left(3^{3}\right)^{5−x}$. Then using Law 3, you can write $3^{4x}=3^{15−3x}$. Since the bases of both sides are equal, you can say that $4x=15−3x$. Rearranging gives $x=15/7$. |

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|  | **Example 4**Solve $x^{\frac{5}{3}}+3x^{\frac{2}{3}}=10x^{−\frac{1}{3}}$.If you look at all of the indices in the question, the denominator is $3$ and that is a hint of what you need to multiply by. Multiply by $x^{\frac{1}{3}}$ on both sides of the equation; using Law 1, this gives Using Law 4, you can further simplify this expression to get $x^{2}+3x=10$, and so $x^{2}+3x−10=0$. This is a quadratic equation; factorizing this equation gives you $\left(x+5\right)\left(x−2\right)=0$, then you can get $x=−5$ and $x=2$ as the two possible solutions to this equation. |

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|  | **Example 5**Solve $2^{x+1}⋅3^{x}=72$.Here, you will need to condense the left-hand side into a single base. Using Law 1, you can write that$$2^{x+1}⋅3^{x}=2\left(2^{x}⋅3^{x}\right)=72.$$Next, use Law 7 to combine $2^{x}$ and $3^{x}$ into a single base;$$2\left(2^{x}⋅3^{x}\right)=2\left(\left(2⋅3\right)^{x}\right)=2⋅6^{x}=72.$$Therefore, $6^{x}=36$. Since $36=6^{2}$, it follows that $x=2$. (Or indeed, you could have taken logarithms of both sides to base $6$.) |

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|  | **Example 6**Solve $\sqrt{\left(6x\right)^{3}}=8\sqrt{27}$.First of all, you can use Law 7 to write $\left(6x\right)^{3}=6^{3}⋅x^{3}$. Using Law 10, you can then write the left hand side of this equation as$$\sqrt{\left(6x\right)^{3}}=\sqrt{6^{3}⋅x^{3}}=\sqrt{6^{3}}⋅\sqrt{x^{3}}$$You can now work on $\sqrt{6^{3}}$; using Law 7 and Law 10, you can write$$\sqrt{6^{3}}=\sqrt{2^{3}⋅3^{3}}=\sqrt{8}⋅\sqrt{27}.$$Therefore, the initial equation becomes$$\sqrt{\left(6x\right)^{3}}=\sqrt{8}⋅\sqrt{27}⋅\sqrt{x^{3}}=8\sqrt{27}.$$Dividing both sides by $\sqrt{8}$ (thereby using Laws 2 and 6) and cancelling the $\sqrt{27}$ gives$$\sqrt{x^{3}}=\frac{8}{\sqrt{8}}=\frac{8^{1}}{8^{1/2}}=8^{1/2}=\sqrt{8}.$$Squaring both sides to undo the square root gives $x^{3}=8$, and so $x=2$. |

## Examples involving logarithms

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|  | **Example 7**Solve $e^{3x}=12$.Taking the logarithm of both sides to base $e$ gives you $log\_{e}\left(e^{3x}\right)=log\_{e}\left(12\right)$. Using the definition of logarithms, you can express the equation as $3x=ln\left(12\right)$. Rearranging the equation gives you $x=\frac{ln\left(12\right)}{3}$ and this is your final answer. |

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|  Tip |
| Although $e$ here was treated as some constant, it is actually a very important constant called *Euler’s number*.In addition, there is a special name for $log\_{e}\left(x\right)$; this is the **natural logarithm** of $x$, often written $ln\left(x\right)$.To see more about Euler’s number $e$ and natural logarithms, please read [Guide: Logarithms](logarithms.qmd). |

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|  | **Example 8**Given the equation $6^{x}=3^{x+1}$, solve for $x$.First of all, you can notice that as $6$ is not a power of $3$ (or vice versa) getting these in the same base is difficult. What you can do instead is use the laws of indices to get an expression of the form $a^{x}=b$ and then take logarithms. Using Law 1, you can write that$$6^{x}=3^{x}⋅3^{1}=3\left(3^{x}\right)$$Dividing both sides by $3^{x}$ and using Law 8 gives$$\left(\frac{6^{x}}{3^{x}}\right)=\left(\frac{6}{3}\right)^{x}=3$$and so you are left with $2^{x}=3$. Taking logarithms of both sides to base $2$ gives $x=log\_{2}\left(3\right)$; and this is your final answer! |

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|  | **Example 9**Given the equation $5^{2x}+7\left(5^{x}\right)−30=0$, you are asked to solve for $x$.Start by letting $y=5^{x}$. Then, you can rewrite the equation given as $\left(5^{x}\right)^{2}+7\left(5^{x}\right)−30=0$ using Law 3, which is the same as writing$$y^{2}+7y−30=0.$$Recognizing that this is a quadratic equation (see [Guide: Introduction to quadratic equations](introtoquadratics.qmd) for more), you can use the quadratic formula or otherwise to show that$$y=−10 or y=3.$$As $y=5^{x}$, this means that $5^{x}=−10$ or $5^{x}=3$. By taking the logarithm of both sides to base $5$, you can show that$$x=log\_{5}\left(−10\right) or x=log\_{5}\left(3\right).$$Remember from above that the logarithm of a negative number is not defined, so $x=log\_{5}\left(3\right)$ is the only viable solution. |

The final example uses many of the laws of indices!

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|  | **Example 10**Solve $3^{3x}=5^{x−4}$.This example is similar to Example 8, only with a few extra steps. Once again, as $3$ is not a power of $5$ or vice versa, the strategy is to write the equation in the form $a^{x}=b$ and then take logarithms of both sides.First of all, use Law 1 to write $5^{x−4}=5^{x}⋅5^{−4}$ and Law 3 to write $3^{3x}=\left(3^{3}\right)^{x}$. Since $3^{3}=27$, the equation becomes$$27^{x}=5^{x}⋅5^{−4}$$Dividing both sides by $5^{x}$ gives$$\frac{27^{x}}{5^{x}}=5^{−4}$$Using Law 8 and Law 5, you can write$$\left(\frac{27}{5}\right)^{x}=\frac{1}{5^{4}}=\frac{1}{625}$$Taking logarithms to base $27/5$ on both sides gives $x=log\_{27/5}\left(1/625\right)$, which is your final answer. |

# Quick check problems

1. Solve $3x^{3}⋅5=405$ for $x$.
2. Solve $\left(9x−1\right)^{1/3}=4$ for $x$.
3. Solve $x^{3/2}=\frac{125}{27}$ for $x$.

# Further reading

For more questions on the subject, please go to [Questions: Solving exponential equations](../questions/qs-solvingeqsindices.qmd).

## Version history and licensing

v1.0: initial version created 08/23 by Ritwik Anand, Zheng Chen, and Zoë Gemmell as part of a University of St Andrews STEP project.

* v1.1: edited 04/24 by tdhc.

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