Laws of indices

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Summary

In mathematics, a knowledge of indices is important for an understanding of algebraic processes. The laws of indices enable expressions involving powers to be manipulated more efficiently than writing them out in full.

What are indices?

Indices (sometimes known as **powers**) are used to display how many times a number has been multiplied by itself. They can also be used to represent roots and fractions, such as the square root. The laws of indices make it possible to alter expressions involving powers more quickly than if they were written out whole.

This guide will introduce six properties of indices: multiplication, division, brackets, powers of zero, negative indices and fractional indices.

Indices are used in many areas of science as a way of representing growth. Economics, biology, demographics and many more fields rely on this sort of growth; chemicals, radioactive waste, and sound are all examples of decay.

Definition of an index

The **index** of a variable (or a constant) is a value that represents how many times a variable is multiplied by itself. Indices are also known as **powers** or **exponents**. It is represented in the form

$$a^r = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{r \text{ times}}$$

Here, a is the **base** and r is the **index**. Indices are a shorthand way of writing multiplications of the same number.
As a multiplied by itself once is a, it follows that anything to the power of one is equal to itself $a^1 = a$ and 1 to the power of anything is 1; so $1^r = 1$ for all real numbers r.

Example 1

Suppose you have

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$$

you can write this as '2 to the power of 5'

$$2^5$$

So

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$$

In this example, 2 is the base and 5 is the index.

Laws of indices

In mathematics, you are not restricted to using positive whole numbers for indices. For example, what do these expressions mean: a^{-1} or a^0 or $a^{\frac{1}{2}}$?

To understand these expressions, you can learn rules to help you deal with these strange looking powers.

Law 1: Multiplication

If the two terms have the same base (in this case a) and are multiplied together, then their indices r and s are **added** (in this case $r + s$)

$$a^r \cdot a^s = a^{r+s}$$

For example, suppose you have 3^2 and you want to multiply it by 3^3. That is

$$3^2 \cdot 3^3 = (3 \cdot 3) \cdot (3 \cdot 3 \cdot 3)$$

which is equal to 3^5.
Therefore, using Law 1 you can get

$$3^2 \cdot 3^3 = 9 \cdot 27 = 243 = 3^5 = 3^{2+3}$$

Example 2

Simplify the following expression: $x^2y^3 \cdot x^3y$.
You can treat x and y as two separate terms and you are now simplifying $x^2 \cdot x^3$ and $y^3 \cdot y$. Using Law 1, you can add the indices to get: x^{2+3} and y^{3+1}. Remembering to multiply the x and y terms, you then get

$$x^2y^3 \cdot x^3y = x^{2+3}y^{3+1} = x^5y^4$$

Law 2: Division

If the two terms have the same base (in this case a) and are to be divided, then their indices are **subtracted** (in this case $r - s$)

$$\frac{a^r}{a^s} = a^{r-s}$$

For example suppose you want to divide 2^5 by 2^3.

$$\frac{2^5}{2^3} = \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2}$$

You can now cancel the common factors of 2. Three of the 2's at the bottom can be cancelled out, so you are left with

$$\frac{2 \cdot 2}{1} = 2^2 = 4$$

Therefore, you can get

$$\frac{2^5}{2^3} = 4 = 2^2 = 2^{5-3}$$

Example 3

Simplify the following expression: $\dfrac{x^2y^3}{x^5y}$

You can treat $\frac{x^2y^3}{x^5y}$ as the product of two fractions: $\frac{x^2}{x^5}$ and $\frac{y^3}{y}$. Using Law 2, you can simplify the fractions to get:

$$\frac{x^2}{x^5} = x^{2-5} = x^{-3}$$

and

$$\frac{y^3}{y} = y^{3-1} = y^2$$

Remembering to multiply the fractions, you get

$$\frac{x^2y^3}{x^5y} = x^{2-5}y^{3-1} = x^{-3}y^2$$

You can notice you have a negative index here; more on those later!

Law 3: brackets

To raise a value or variable (in this case a) presented in index form to another index, **multiply** the indices together (r and s)

$$(a^r)^s = a^{r \cdot s}$$

For example suppose you had 3^2 and you want to raise it all to the power of 3; that is $(3^2)^3$, which means

$$3^2 \cdot 3^2 \cdot 3^2.$$

Law 1 tells you to add the indices together. So you can get

$$3^6$$

But note that 6 is $2 \cdot 3$. This suggests that Law 3 also works. Therefore, using Law 3 you can get

$$(3^2)^3 = 729 = 3^6 = 3^{2 \cdot 3}$$

Example 4

Simplify the following expression: $(x^2y^3)^4$
Using Law 3, you get

$$(x^2y^3)^4 = (x^2)^4 \cdot (y^3)^4 = x^{2\cdot 4}y^{3\cdot 4} = x^8y^{12}$$

Law 4: power of 0

Any non-zero real number a to the power of zero is equal to one:

$$a^0 = 1$$

Warning

0^0 is indeterminate, anything to the power of 0 is 1, but 0 to the power of anything is 0, so it is not defined.

Example 5

Simplify the following expression: $(x^2 + y^5)^0$
Using Law 4, you can get

$$(x^2 + y^5)^0 = 1$$

Law 5: negative indices

If the index is a negative value, then it is reciprocal of the positive index raised to the same variable

$$a^{-r} = \frac{1}{a^r}$$

For example, suppose you want to divide 2^3 by 2^7

$$\frac{2^3}{2^7} = \frac{2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$$

Again, you can find the common factors of 2. The three 2's at the top can be cancelled out, so you are left with

$$\frac{2^3}{2^7} = \frac{1}{2 \cdot 2 \cdot 2 \cdot 2} = \frac{1}{2^4}$$

Now consider Law 2, you can do the similar calculation by subtracting the indices

$$\frac{2^3}{2^7} = 2^{3-7} = 2^{-4}$$

You have done the calculation in two different ways. So you get

$$\frac{2^3}{2^7} = 2^{3-7} = 2^{-4} = \frac{1}{2^4}$$

Example 6

Simplify the following expression: $(3x + 4y)^{-1}$
Using Law 5, you can get

$$(3x + 4y)^{-1} = \frac{1}{3x + 4y}$$

Law 6: fractional indices

An index in the form $1/n$ (where n is a positive whole number) is an nth root of a. So

$$a^{1/n} = \sqrt[n]{a}.$$

For example: suppose you found that

$$2^r \cdot 2^r = 2$$

This means that multiplying 2^r by 2^r gives the result 2. You know that $2 = 2^1$, so consider Law 1, $2^r \cdot 2^r = 2^{2r}$. If $2^r \cdot 2^r = 2$ then you can get

$$2^{2r} = 2^1$$

from which $r = 1$; so $r = \frac{1}{2}$.This shows that

$$2^{\frac{1}{2}} = \sqrt[2]{2} = \sqrt{2}$$

What about an expression of the form $a^{r/n}$, where r is a real number and n is a positive whole number? Combining Laws 3 and 6 yields the following:

$$a^{r/n} = (a^{1/n})^r = (\sqrt[n]{a})^r$$

or equivalently $a^{r/n} = (a^r)^{1/n}$.

Example 7

(a) Simplify $10000^{\left(\frac{3}{4}\right)}$.

Using Law 7, you can re-express $10000^{\frac{3}{4}}$ as $(\sqrt[4]{10000})^3$. Since $10^4 = 10000$, it follows that $\sqrt[4]{10000} = 10$. Finally, you can say that $(\sqrt[4]{10000})^3 = 10^3 = 1000$.

(b) Simplify $27^{-\frac{2}{3}}$.

Using Law 7, you can write this expression as $27^{-\frac{2}{3}} = (\sqrt[3]{27})^{-2}$. Since the cube root of 27 is 3, it follows that

$$(\sqrt[3]{27})^{-2} = 3^{-2}$$

which, using Law 5, becomes $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$.

The first six laws above focused on the index and when the index changes; and in this case, the base remains constant. The final four laws focus on when the base changes and the index remains constant.

This next law follows from the fact that multiplication is *commutative*; so $ab = ba$ for any two real numbers a, b.

Law 7: multiplication of variables with the same indices but different bases

When multiplying variables with the same powers you can expand and rewrite them into a nicer format:

$$a^r \cdot b^r = (ab)^r$$

Where the bases are multiplied together then raised to the rth power. For example

$$2^3 \cdot 3^3 = \underbrace{(2 \cdot 2 \cdot 2)}_{\text{3 times}} \cdot \underbrace{(3 \cdot 3 \cdot 3)}_{\text{3 times}} = (2 \cdot 3) \cdot (2 \cdot 3) \cdot (2 \cdot 3) = (2 \cdot 3)^3.$$

Example 8

(a) Here, you can use Law 7 to say that $(3x)^3 = 3^3 \cdot x^3 = 27x^3$. Notice that $(3x)^3$ is **not** necessarily equal to $3x^3$!

(b) You can use Law 7 in conjunction with other laws. For instance, how would you simplify $(x^3y^2z)^{-2}$? Using Law 7 twice, you can write

$$(x^3y^2z)^{-2} = (x^3)^{-2} \cdot (y^2)^{-2} \cdot z^{-2}$$

Now, using Law 3, you can write

$$(x^3)^{-2} \cdot (y^2)^{-2} \cdot z^{-2} = x^{-6} \cdot y^{-4} \cdot z^{-2}$$

Finally, using Law 5 you can say that

$$(x^3y^2z)^{-2} = x^{-6} \cdot y^{-4} \cdot z^{-2} = \frac{1}{x^6} \cdot \frac{1}{y^4} \cdot \frac{1}{z^2} = \frac{1}{x^6y^4z^2}.$$

Law 8: division of variables with the same indices but different bases

When dividing variables with the same powers you can expand and rewrite them into a nicer format:

$$\frac{a^r}{b^r} = \left(\frac{a}{b}\right)^r$$

Where the numerator base a is divided by the denominator base b then raised to the rth power. For example:

$$\frac{4^4}{5^4} = \frac{\overbrace{4 \cdot 4 \cdot 4 \cdot 4}^{4\text{ times}}}{\underbrace{5 \cdot 5 \cdot 5 \cdot 5}_{4\text{ times}}} = \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} = \left(\frac{4}{5}\right)^4$$

Example 9

Here, you can use Law 8 (and Laws 3 and 7) to simplify $(2x/y^2)^3$ in the following way:

$$\left(\frac{2x}{y^2}\right)^3 = \frac{(2x)^3}{(y^2)^3} = \frac{8x^3}{y^6}.$$

Law 9: fractional power, the nth root

When multiplying by bases with fractional powers (some sort of nth root) :

$$a^{1/n} \cdot b = \sqrt[n]{a \cdot b^n}$$

Where the aim of the manipulation is to put everything inside the nth root sign, then to continue with the calculation.

The main purpose of this law is to combine bases if you can; as evidenced in the next example.

Example 10

Simplify $2^{1/3} \cdot 4$. Here, you can use Law 9 to write

$$2^{1/3} \cdot 4 = \sqrt[3]{2 \cdot 4^3}.$$

You know that $4 = 2^2$, and so using Law 3 and then Law 1 gives

$$\sqrt[3]{2 \cdot 4^3} = \sqrt[3]{2 \cdot (2^2)^3} = \sqrt[3]{2 \cdot 2^6} = \sqrt[3]{2^7}$$

Finally, using Law 6 gives

$$2^{1/3} \cdot 4 = \sqrt[3]{2^7} = 2^{7/3}.$$

Law 10: multiplication under the root sign

When you have a term under a nth root sign which you can factorise (split up into different factors) this can be done and the nth root signs can be dealt with separately.

$$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

Where you are using the commutativity of multiplication to split up the nth root and deal with terms separately.

This law is particularly useful for writing square roots more concisely. For example

$$\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$$

Example 11

Simplify $\sqrt[3]{27x^9}$. First of all, you can use Law 10 to say that

$$\sqrt[3]{27x^9} = \sqrt[3]{27} \cdot \sqrt[3]{x^9}.$$

You know that $\sqrt[3]{27} = 3$; using Law 6 and Law 3 gives

$$\sqrt[3]{27x^9} = \sqrt[3]{27} \cdot \sqrt[3]{x^9} = 3(x^9)^{1/3} = 3x^{9/3} = 3x^3.$$

Quick check problems

Determine whether the following equations are correct or not:

(a) $\dfrac{17x^{17}y^{14}}{15x^{18}y^{18}} = \dfrac{17}{15xy}$

(b) $(5x^2 \cdot 2x^5 \cdot 3x^4)^2 = 900x^{22}$

(c) $\dfrac{12x^{14}y^{-4}}{12x^7y^{11}} = \dfrac{x^6}{y^7}$

(d) $\dfrac{3x^a}{y^{8b}} \cdot \dfrac{4z^a}{y^d} = \dfrac{12(xz)^a}{y^{d+8b}}$

(e) $\dfrac{4^2}{2^4} = 2$

Further reading

For more questions on the subject, please go to Questions: Laws of indices

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