Introduction to algebraic fractions

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Summary

Algebraic fractions extend the idea of numerical fractions to expressions that include variables. This guide explains what algebraic fractions are and why certain values are not allowed in the denominator. You will learn how to rewrite algebraic fractions in equivalent forms and how to simplify them by factorizing and cancelling common factors. These skills are essential for later topics such as solving algebraic equations, manipulating formulas, and working with rational functions.

Before reading this guide, it is recommended that you read Guide: Introduction to fractions, Guide: Arithmetic on fractions and Guide: Factorization.

What is an algebraic fraction?

In Guide: Introduction to fractions, you learned about numerical fractions, which are fractions where both the numerator and denominator are numbers:

$$\frac{3}{4} \qquad \frac{15}{7} \qquad \frac{23}{100}$$

These ideas can be extended to **algebraic fractions**. These look similar, but instead of including only numbers, the numerator or denominator (or both) can include letters and expressions:

$$\frac{x}{5} \qquad \frac{2}{x^2} \qquad \frac{x+3}{x-2} \qquad \frac{4\pi r^3}{3} \qquad \frac{1}{x^2+y^2}$$

By the end of this guide, you will be able to find equivalent fractions for algebraic fractions and simplify them in a similar way to numerical fractions. The main difference between algebraic and numerical fractions is that, because the denominator may contain algebra, you must be careful about dividing by zero.

Restrictions on the denominator

Every algebraic fraction has a denominator, and one of the most important rules in mathematics is that **you can never divide by zero**.

If the denominator of a fraction becomes zero for a certain value of a variable, then the fraction is said to be **undefined** at that point. This means that the expression has no value at that point.

> **i** Restrictions on the denominator
>
> When working with algebraic fractions, you need to find any values of the variable(s) that make the denominator zero. These are called the **restricted values**. Any condition that says that a variable is not equal to a restricted value is called a **restriction**.

For instance, for the fraction

$$\frac{1}{x-1}$$

- $x = 1$ is a **restricted value**
- $x \neq 1$ is a **restriction**.

> **!** Important
>
> You should clearly state the restriction for any algebraic fraction you use.

> **i** **Example 1**
>
> For what values of x is the following fraction undefined?
>
> $$\frac{2}{x}$$
>
> Here, the denominator is x. The fraction is undefined when $x = 0$, so the restriction on this expression is $x \neq 0$.

> **i** **Example 2**
>
> For what values of x is the following fraction undefined?
>
> $$\frac{1}{x-3}$$
>
> Here, the denominator is $x - 3$. The fraction is undefined when $x - 3 = 0$, which occurs when $x = 3$. So the restriction is $x \neq 3$.

Example 3

For what values of x is the following fraction undefined?

$$\frac{x+2}{(x-1)(x+4)}$$

Here the denominator is $(x-1)(x+4)$. The fraction is undefined when $(x-1)(x+4)=0$, which occurs when either $x=1$ or $x=-4$, so the restrictions are

$$x \neq 1 \text{ and } x \neq -4.$$

Tip

Restrictions of multiple values can also be written in set notation. In this example, you can also write $x \notin \{1, -4\}$. The symbol $\in$ means "is a member of" and the symbol $\notin$ means "is not a member of". This means that $x \notin \{1, -4\}$ reads as "x is not any of 1 and -4". For more on this, see [Guide: Introduction to sets].

Example 4

For what values of x and y is the following fraction undefined?

$$\frac{x+2y}{xy}$$

Here the denominator is xy. The fraction is undefined when $xy=0$, which occurs when $x=0$, or $y=0$, or when both $x=y=0$. So the restrictions are

$$x \neq 0 \text{ and } y \neq 0,$$

which you can also write as $x, y \neq 0$.

Example 5

For what values of x and y is the following fraction undefined?

$$\frac{10}{x^2 + y^2}$$

Here the denominator is $x^2 + y^2$. The fraction is undefined when $x^2 + y^2 = 0$. Since $x^2, y^2 > 0$ for all x and y not equal to zero, it follows that $x^2 + y^2 = 0 $ occurs only when both x and y are 0. So the restrictions are

$$x \neq 0 \text{ or } y \neq 0,$$

as $x^2 + y^2 \neq 0$ whenever one or both of x or y are non-zero.

Tip

Notice the difference between the two denominators xy and $x^2 + y^2$ in Examples 4 and 5. In Example 4 for xy, the restricted values were $x \neq 0$ **and** $y \neq 0$; in Example 5 for $x^2 + y^2$, the restricted values were $x \neq 0$ **or** $y \neq 0$. This is a crucial difference, and is a way into thinking **logically** about your work. For more on this, see [Guide: Introduction to predicate logic].

Equivalent algebraic fractions

Similarly to numerical fractions, two algebraic fractions can look different but still represent the same value. If you multiply or divide both the numerator and the denominator by the same non-zero expression, the fraction's value does not change. In this case, what you have done is written it in a new form.

Equivalent algebraic fractions are important because they let you rewrite fractions with different denominators when adding or subtracting.

Equivalent algebraic fractions

Two algebraic fractions are equivalent if they have the same value, even though the expressions may look different.
You can create an equivalent algebraic fraction by:

- Multiplying the numerator and denominator by the same **non-zero** expression.
- Dividing the numerator and denominator by the same common factor.

Both of these are the same thing as multiplying your expression by 1.

! Important

Dividing the numerator and denominator by the same common factor is known as **cancelling the factors out**, or even only **cancelling**.

Example 4

Below are some examples of equivalent algebraic fractions.
Multiplying both numerator and denominator by the same non-zero constant:

$$\frac{1}{x} = \frac{1 \cdot 3}{x \cdot 3} = \frac{3}{3x} \qquad \text{if } x \neq 0$$

Dividing both numerator and denominator by a common factor (**cancelling**):

$$\frac{6x^2}{9x} = \frac{6x^2 \div 3x}{9x \div 3x} = \frac{2x}{3} \qquad \text{if } x \neq 0$$

Multiplying both numerator and denominator by the same non-zero variable expression:

$$\frac{2}{x} = \frac{2(t+1)}{x(t+1)} \qquad \text{if } x \neq 0 \text{ and } t \neq -1$$

Writing a non-zero whole number as an algebraic fraction:

$$2 = \frac{2}{1} = \frac{2x}{x} = \frac{2x(x+1)}{x(x+1)} \qquad \text{if } x \neq 0 \text{ and } x \neq -1$$

Simplifying algebraic fractions

Similarly to numerical fractions, algebraic fractions can also be written in a simplified form. Simplifying makes expressions more convenient to work with, and it helps when comparing fractions or solving equations.

Simplifying algebraic fractions

A **simplified algebraic fraction** is one where you have divided the numerator and denominator by all their common factors.
To simplify an algebraic fraction:

1. Factorize the numerator and the denominator completely.

2. State the restrictions on the denominator, if any exist.
3. Cancel any common factors that appear in both the numerator and the denominator.
4. Write the final simplified result.

Important

Between the numerator and the denominator, you can only cancel **factors**, not terms.

- A factor is part of a multiplication. In $x(x+3)$, both x and $(x+3)$ are factors.
- A term is part of an addition or subtraction. In $x+3$, both x and 3 are terms.

This is why, in the simplifying process, you cannot cancel the x terms by writing

$$\frac{x+3}{x+2} = \frac{\not{x}+3}{\not{x}+2} = \frac{3}{2}$$

For more about factors and terms, see Guide: Factorization.

Important

It's really important that the restrictions are kept, even when the simplified algebraic expression doesn't require any. This is because when cancelling factors, you cannot cancel by a factor of 0 on top and bottom. In addition to this, you can't 'forget' that the restrictions ever existed in the first place. This is of critical importance when it comes to viewing algebraic fractions as formal functions with domains and codomains: see [Guide: Introduction to real-valued functions] for more.

Example 6

Simplify $\frac{6x}{9x}$.
Both the numerator and denominator contain x as a factor. **As long as** $x \neq 0$, it can be cancelled out to reduce the algebraic fraction to a numerical fraction.

$$\frac{6x}{9x} = \frac{6}{9} = \frac{2}{3} \qquad \text{if } x \neq 0$$

Example 7

Simplify $\dfrac{x^2+3x}{x}$.
The numerator can be factorized to give a common factor of x:

$$x^2+3x = x(x+3)$$

As long as $x \neq 0$, the common factor of x can be cancelled out to give $x+3$.

$$\frac{x^2+3x}{x} = \frac{x(x+3)}{x} = x+3 \qquad \text{if } x \neq 0$$

Example 8

Simplify $\dfrac{x^2-9}{x^2-3x}$.
The numerator can be factorized as it is a difference of two squares:

$$x^2-9 = (x+3)(x-3)$$

The denominator can be factorized to give a common factor of x:

$$x^2-3x = x(x-3)$$

As long as $x \neq 3$, the common factor of $x-3$ can be cancelled out to give $\frac{x+3}{x}$.

$$\frac{x^2-9}{x^2-3x} = \frac{(x+3)(x-3)}{x(x-3)} = \frac{x+3}{x} \qquad \text{if } x \neq 0 \text{ and } x \neq 3$$

Here's one final example that brings together everything you have learned.

Example 9

You are given the fraction

$$\frac{tx + ty}{x^2 - y^2}$$

Give its restricted values and simplify as much as you can.
First of all, the restricted values are when $x^2 - y^2 = 0$. This happens when $x^2 = y^2$. As negative numbers squared equals a positive number, it follows that $x^2 = y^2$ when $x = y$ or $x = -y$. You can write $x = \pm y$ to represent this. It follows that the fraction is defined when $x \neq pmy$.
Now, how can you simplify this?
The numerator has a common factor of t, so you can factorize to get :

$$tx + ty = t(x + y)$$

The denominator can be factorized as it is a difference of two squares

$$x^2 - y^2 = (x + y)(x - y)$$

As long as $x \neq -y$, the common factor of $x + y$ can be cancelled out to give $\frac{t}{x-y}$.
So

$$\frac{tx + ty}{x^2 - y^2} = \frac{t(x + y)}{(x + y)(x - y)} = \frac{t}{x - y} \qquad \text{if } x \neq \pm y.$$

Quick check problems

1. For which value of x is the fraction $\frac{3}{x-2}$ undefined?

(a) $x = 0$

(b) $x = 2$

(c) $x = -2$

(d) There are no restricted values

2. Which of the following fractions is equivalent to $\frac{4}{x}$?

(a) $\frac{4x}{x^2}$

(b) $\frac{8}{4x}$

(c) $\dfrac{4}{x+1}$

(d) $\dfrac{x}{4}$

3. Simplify the fraction $\frac{6x^2}{3x}$, assuming $x \neq 0$.

4. For which value(s) of x is the fraction $\frac{x+5}{x(x-4)}$ undefined?

(a) $x = 0$ only

(b) $x = 4$ only

(c) $x = 0$ or $x = 4$

(d) There are no restricted values

Further reading

For more questions on the subject, please go to Questions: Introduction to algebraic fractions.

To learn how to perform arithmetic on algebraic fractions, please see Guide: Arithmetic on algebraic fractions.

Version history

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