Proof: Rules of differentiation

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Summary

The proof sheet demonstrates that the common rules of differentiation - the sum, difference, product, quotient, and chain rules - are true.

*Before reading this proof sheet, it is essential that you read* [*Guide: Introduction to differentiation and the derivative*](../studyguides/introtodifferentiation.qmd)*. In addition, reading [Guide: Introduction to limits] is useful. Further reading will be illustrated where required.*

The starting point of this proof sheet is the limit definition of the derivative of a function:

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|  Reminder of limit definition of the derivative |
| The **derivative of** $f\left(x\right)$ **with respect to** $x$ is defined to be the function$$f′\left(x\right)=\lim\_{h\to 0}\frac{f\left(x+h\right)−f\left(x\right)}{h}.$$ |

# Sum and difference rules

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|  The sum and difference rules |
| (**sum rule**) The derivative of two functions $f\left(x\right)$ and $g\left(x\right)$ added together is the same as their derivatives $f′\left(x\right)$ and $g′\left(x\right)$ added together; that is, $\left(f+g\right)′\left(x\right)=f′\left(x\right)+g′\left(x\right)$ or$$\frac{d}{dx}\left(f+g\right)=\frac{df}{dx}+\frac{dg}{dx}$$(**difference rule**) The derivative of one function $g\left(x\right)$ subtracted from another $f\left(x\right)$ is the same as the derivative $g′\left(x\right)$ subtracted from the derivative of $f′\left(x\right)$; that is $\left(f−g\right)′\left(x\right)=f′\left(x\right)−g′\left(x\right)$ or$$\frac{d}{dx}\left(f−g\right)=\frac{df}{dx}−\frac{dg}{dx}$$ |

## Proof of the sum rule

The strategy here is direct; put the function $\left(f+g\right)$ into the definition and pull the fraction apart to reveal the definitions of derivatives of $f$ and $g$.

Let’s start with $f\left(x\right)$ and $g\left(x\right)$ as two differentiable real-valued functions, with sum $\left(f+g\right)\left(x\right)=f\left(x\right)+g\left(x\right)$. Putting this into the limit definition of the derivative given above:

$$\left(f+g\right)′\left(x\right)=\lim\_{h\to 0}\frac{\left(f+g\right)\left(x+h\right)−\left(f+g\right)\left(x\right)}{h}$$

Since $\left(f+g\right)\left(x\right)=f\left(x\right)+g\left(x\right)$, this becomes

$$\begin{matrix}\left(f+g\right)′\left(x\right)&=\lim\_{h\to 0}\frac{f\left(x+h\right)+g\left(x+h\right)−\left(f\left(x\right)+g\left(x\right)\right)}{h}\\&=\lim\_{h\to 0}\frac{f\left(x+h\right)+g\left(x+h\right)−f\left(x\right)−g\left(x\right)}{h}\end{matrix}$$

You can now split this into two fractions, one of which sets up the definition of $f′\left(x\right)$, and the other sets up the definition of $g′\left(x\right)$. So here

$$\begin{matrix}\left(f+g\right)′\left(x\right)&=\lim\_{h\to 0}\frac{f\left(x+h\right)+g\left(x+h\right)−f\left(x\right)−g\left(x\right)}{h}\\&=\lim\_{h\to 0}\left(\frac{f\left(x+h\right)−f\left(x\right)}{h}+\frac{g\left(x+h\right)−g\left(x\right)}{h}\right)\end{matrix}$$

Now, use properties of limits (see [Guide: Introduction to limits]) to split the limits gives

$$\begin{matrix}\left(f+g\right)′\left(x\right)&=\lim\_{h\to 0}\left(\frac{f\left(x+h\right)−f\left(x\right)}{h}+\frac{g\left(x+h\right)−g\left(x\right)}{h}\right)\\&=\lim\_{h\to 0}\frac{f\left(x+h\right)−f\left(x\right)}{h}+\lim\_{h\to 0}\frac{g\left(x+h\right)−g\left(x\right)}{h}\\\end{matrix}$$

and so, by the limit definition of the derivative

$$\left(f+g\right)′\left(x\right)=\lim\_{h\to 0}\frac{f\left(x+h\right)−f\left(x\right)}{h}+\lim\_{h\to 0}\frac{g\left(x+h\right)−g\left(x\right)}{h}=f′\left(x\right)+g′\left(x\right)$$

as required.

## Proof of the difference rule

Let’s start with $f\left(x\right)$ and $g\left(x\right)$ as two differentiable real-valued functions, with difference $\left(f−g\right)\left(x\right)=f\left(x\right)−g\left(x\right)$. Putting this into the limit definition of the derivative given above:

$$\left(f−g\right)′\left(x\right)=\lim\_{h\to 0}\frac{\left(f−g\right)\left(x+h\right)−\left(f−g\right)\left(x\right)}{h}$$

Using the fact that $\left(f−g\right)\left(x\right)=f\left(x\right)−g\left(x\right)$, and taking care of the signs in expansion, gives

$$\begin{matrix}\left(f−g\right)′\left(x\right)&=\lim\_{h\to 0}\frac{f\left(x+h\right)−g\left(x+h\right)−\left(f\left(x\right)−g\left(x\right)\right)}{h}\\&=\lim\_{h\to 0}\frac{f\left(x+h\right)−g\left(x+h\right)−f\left(x\right)+g\left(x\right)}{h}\end{matrix}$$

You can now split this into two fractions, one of which sets up the definition of $f′\left(x\right)$, and the other sets up the definition of $g′\left(x\right)$. So here

$$\begin{matrix}\left(f−g\right)′\left(x\right)&=\lim\_{h\to 0}\frac{f\left(x+h\right)−g\left(x+h\right)−f\left(x\right)+g\left(x\right)}{h}\\&=\lim\_{h\to 0}\left(\frac{f\left(x+h\right)−f\left(x\right)}{h}−\frac{g\left(x+h\right)−g\left(x\right)}{h}\right)\end{matrix}$$

Now, use properties of limits (see [Guide: Introduction to limits]) to split the limits gives

$$\begin{matrix}\left(f−g\right)′\left(x\right)&=\lim\_{h\to 0}\left(\frac{f\left(x+h\right)−f\left(x\right)}{h}−\frac{g\left(x+h\right)−g\left(x\right)}{h}\right)\\&=\lim\_{h\to 0}\frac{f\left(x+h\right)−f\left(x\right)}{h}−\lim\_{h\to 0}\frac{g\left(x+h\right)−g\left(x\right)}{h}\\\end{matrix}$$

and so, by the limit definition of the derivative

$$\left(f−g\right)′\left(x\right)=\lim\_{h\to 0}\frac{f\left(x+h\right)−f\left(x\right)}{h}−\lim\_{h\to 0}\frac{g\left(x+h\right)−g\left(x\right)}{h}=f′\left(x\right)−g′\left(x\right)$$

as required.

# Scaling rule

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|  The scaling rule |
| The derivative of a function $f\left(x\right)$ multiplied by a real number $c$ is the same as the derivative $f′\left(x\right)$ multiplied by $c$; that is $\left(cf\right)′\left(x\right)=cf′\left(x\right)$ or$$\frac{d}{dx}\left(cf\left(x\right)\right)=c\frac{df}{dx}$$ |

## Proof of the scaling rule

This is similar to that of the sum and difference rules. Let’s start with $f\left(x\right)$ as a differentiable real-valued function, with scaling $\left(cf\right)\left(x\right)=cf\left(x\right)$. Putting this into the limit definition of the derivative given above:

$$\left(cf\right)′\left(x\right)=\lim\_{h\to 0}\frac{\left(cf\right)\left(x+h\right)+\left(cf\right)\left(x\right)}{h}$$

Using the fact that $\left(cf\right)\left(x\right)=cf\left(x\right)$ and factorizing out the $c$ gives

$$\begin{matrix}\left(cf\right)′\left(x\right)&=\lim\_{h\to 0}\frac{\left(cf\right)\left(x+h\right)−\left(cf\right)\left(x\right)}{h}\\&=\lim\_{h\to 0}\frac{cf\left(x+h\right)−cf\left(x\right)}{h}=\lim\_{h\to 0}\frac{c\left(f\left(x+h\right)−f\left(x\right)\right)}{h}\end{matrix}$$

Since the constant $c$ does not depend on the variable in the limit $h$, you can use properties of limits (see [Guide: Introduction to limits]) to take the constant $c$ out of the limit. This gives

$$\begin{matrix}\left(cf\right)′\left(x\right)&=\lim\_{h\to 0}\frac{c\left(f\left(x+h\right)−f\left(x\right)\right)}{h}\\&=c⋅\lim\_{h\to 0}\frac{f\left(x+h\right)−f\left(x\right)}{h}\\\end{matrix}$$

and so, by the limit definition of the derivative

$$\left(cf\right)′\left(x\right)=c⋅\lim\_{h\to 0}\frac{f\left(x+h\right)−f\left(x\right)}{h}=cf′\left(x\right)$$

as required.

# Product rule

*See* [*Guide: The product rule*](productrule.qmd) *for more about the product rule.*

Here is the product rule, restated with $f\left(x\right)=u\left(x\right)$ and $g\left(x\right)=v\left(x\right)$ for visual ease in the proof that follows.

|  |
| --- |
|  The product rule |
| Let $f\left(x\right)$ and $g\left(x\right)$ be two differentiable functions. Then the **product rule** says that$$\left(fg\right)′\left(x\right)=\frac{d}{dx}\left(f\left(x\right)g\left(x\right)\right)=f\left(x\right)g′\left(x\right)+f′\left(x\right)g\left(x\right)$$that is, the derivative of the product of $f\left(x\right)$ and $g\left(x\right)$ is equal to the product of $f\left(x\right)$ and the derivative of $g\left(x\right)$, plus the product of $g\left(x\right)$ and the derivative of $f\left(x\right)$.This can also be written as$$\frac{d}{dx}\left(f\left(x\right)g\left(x\right)\right)=f\frac{dg}{dx}+g\frac{df}{dx}.$$ |

## Proof of the product rule

Here’s why the product rule works. It requires a little more thought than the proof of the sum rule and the scaling rule; you have to manufacture the definition of the derivative in one place by using a creative addition by $0$.

So, let’s start with $f\left(x\right)$ and $g\left(x\right)$ as two differentiable real-valued functions, with product $\left(fg\right)\left(x\right)=f\left(x\right)g\left(x\right)$. Putting this into the limit definition of the derivative given above:

$$\left(fg\right)′\left(x\right)=\lim\_{h\to 0}\frac{\left(fg\right)\left(x+h\right)−\left(fg\right)\left(x\right)}{h}$$

Since $\left(fg\right)\left(x\right)=f\left(x\right)g\left(x\right)$, this becomes

$$\left(fg\right)′\left(x\right)=\lim\_{h\to 0}\frac{f\left(x+h\right)g\left(x+h\right)−f\left(x\right)g\left(x\right)}{h}$$

Now, there’s no way of pulling this apart. You have to force the issue slightly by creatively adding $0$. The way to do this is to add $−f\left(x+h\right)g\left(x\right)+f\left(x+h\right)g\left(x\right)$ into the numerator, and factorize in slightly different ways. This is fine to do, as $−f\left(x+h\right)g\left(x\right)+f\left(x+h\right)g\left(x\right)=0$. Doing this, and factorizing to manufacture the definitions of $f′\left(x\right)$ and $g′\left(x\right)$ gives:

$$\begin{matrix}\left(fg\right)′\left(x\right)&=\lim\_{h\to 0}\frac{f\left(x+h\right)g\left(x+h\right)−f\left(x\right)g\left(x\right)}{h}\\&=\lim\_{h\to 0}\frac{f\left(x+h\right)g\left(x+h\right)−f\left(x+h\right)g\left(x\right)+f\left(x+h\right)g\left(x\right)−f\left(x\right)g\left(x\right)}{h}\\&=\lim\_{h\to 0}\left(f\left(x+h\right)\frac{g\left(x+h\right)−g\left(x\right)}{h}+g\left(x\right)\frac{f\left(x+h\right)−f\left(x\right)}{h}\right)\end{matrix}$$

Using properties of limits, and the fact that $g\left(x\right)$ is constant as $h$ varies to take it outside the limit gives

$$\begin{matrix}\left(fg\right)′\left(x\right)&=\lim\_{h\to 0}\left(f\left(x+h\right)\frac{g\left(x+h\right)−g\left(x\right)}{h}+g\left(x\right)\frac{f\left(x+h\right)−f\left(x\right)}{h}\right)\\&=\left(\lim\_{h\to 0}f\left(x+h\right)\right)\left(\lim\_{h\to 0}\frac{g\left(x+h\right)−g\left(x\right)}{h}\right)+g\left(x\right)\left(\lim\_{h\to 0}\frac{f\left(x+h\right)−f\left(x\right)}{h}\right)\end{matrix}$$

Now, as $h$ tends to $0$, it follows that $f\left(x+h\right)$ tends to $f\left(x\right)$. The other two limits are the definitions of $g′\left(x\right)$ and $f′\left(x\right)$ respectively. Therefore, you can write that

$$\begin{matrix}\left(fg\right)′\left(x\right)&=\left(\lim\_{h\to 0}f\left(x+h\right)\right)\left(\lim\_{h\to 0}\frac{g\left(x+h\right)−g\left(x\right)}{h}\right)+g\left(x\right)\left(\lim\_{h\to 0}\frac{f\left(x+h\right)−f\left(x\right)}{h}\right)\\&=f\left(x\right)g′\left(x\right)+g\left(x\right)f′\left(x\right)\end{matrix}$$

which is the product rule.

# Quotient rule

*See* [*Guide: The quotient rule*](quotientrule.qmd) *for more about the quotient rule.*

Here is the quotient rule, restated with $f\left(x\right)=u\left(x\right)$ and $g\left(x\right)=v\left(x\right)$ for visual ease in the proof that follows.

|  |
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|  The quotient rule |
| Let $f\left(x\right)$ and $g\left(x\right)$ be two differentiable functions. Then the **quotient rule** says that$$\left(\frac{f}{g}\right)′\left(x\right)=\frac{d}{dx}\left(\frac{f\left(x\right)}{g\left(x\right)}\right)=\frac{f′\left(x\right)g\left(x\right)−f\left(x\right)g′\left(x\right)}{\left(g\left(x\right)\right)^{2}}$$that is, the derivative of $u\left(x\right)$ divided by $v\left(x\right)$ is equal to the difference of $u′\left(x\right)v\left(x\right)$ and $u\left(x\right)v′\left(x\right)$, divided by the square of the function $v\left(x\right)$.This can also be written as$$\frac{d}{dx}\left(\frac{u\left(x\right)}{v\left(x\right)}\right)=\frac{v\frac{du}{dx}−u\frac{dv}{dx}}{v^{2}}.$$ |

## Proof of the quotient rule

Here’s why the quotient rule works. Again, there is a step beyond algebraic manipulation where you have to manufacture the definition of the derivative in one place by using a creative addition by $0$.

So, let’s start with $f\left(x\right)$ and $g\left(x\right)$ as two differentiable real-valued functions (with $g\left(x\right)$ not the zero function), with quotient $\left(f/g\right)\left(x\right)=f\left(x\right)/g\left(x\right)$. Putting this into the limit definition of the derivative gives

$$\left(\frac{f}{g}\right)′\left(x\right)=\lim\_{h\to 0}\frac{\left(\frac{f}{g}\right)\left(x+h\right)−\left(\frac{f}{g}\right)\left(x\right)}{h}=\lim\_{h\to 0}\frac{\frac{f\left(x+h\right)}{g\left(x+h\right)}−\frac{f\left(x\right)}{g\left(x\right)}}{h}$$

You can try your best to reduce this down by cross-multiplying to get a common denominator of the numerator of the limit. Then, you can drop that denominator down to get a single fraction. Doing this:

$$\begin{matrix}\left(\frac{f}{g}\right)′\left(x\right)&=\lim\_{h\to 0}\frac{\frac{f\left(x+h\right)}{g\left(x+h\right)}−\frac{f\left(x\right)}{g\left(x\right)}}{h}\\&=\lim\_{h\to 0}\frac{\frac{f\left(x+h\right)g\left(x\right)−f\left(x\right)g\left(x+h\right)}{g\left(x\right)g\left(x+h\right)}}{h}\\&=\lim\_{h\to 0}\frac{f\left(x+h\right)g\left(x\right)−f\left(x\right)g\left(x+h\right)}{hg\left(x\right)g\left(x+h\right)}\end{matrix}$$

Now, the hope is to pull this apart into two separate limits. Since you have no way of cancelling the $h$, you could try and manufacture the definitions of the derivatives of $f\left(x\right)$ and $g\left(x\right)$. You have to force the issue slightly by creatively adding $0$; in this case, by adding $−f\left(x\right)g\left(x\right)+f\left(x\right)g\left(x\right)=0$ to the numerator. In addition, you can use properties of limits to get rid of the $g\left(x\right)g\left(x+h\right)$ in the denominator. Doing these steps and simplifying gives:

$$\begin{matrix}\left(\frac{f}{g}\right)′\left(x\right)&=\lim\_{h\to 0}\frac{f\left(x+h\right)g\left(x\right)−f\left(x\right)g\left(x+h\right)}{hg\left(x\right)g\left(x+h\right)}\\&=\left(\lim\_{h\to 0}\frac{f\left(x+h\right)g\left(x\right)−f\left(x\right)g\left(x+h\right)}{h}\right)\left(\lim\_{h\to 0}\frac{1}{g\left(x\right)g\left(x+h\right)}\right)\\&=\left(\lim\_{h\to 0}\frac{f\left(x+h\right)g\left(x\right)−f\left(x\right)g\left(x\right)+f\left(x\right)g\left(x\right)−f\left(x\right)g\left(x+h\right)}{h}\right)\left(\lim\_{h\to 0}\frac{1}{g\left(x\right)g\left(x+h\right)}\right)\end{matrix}$$

Now, factorizing this expression, using the properties of limits) and moving $g\left(x\right)$ and $−f\left(x\right)$ (notice that this needs to be done to ensure the correct definition of the derivative) out of the limits where appropriate gives

$$\begin{matrix}\left({f}/{g}\right)′\left(x\right)&=\left(\lim\_{h\to 0}{f\left(x+h\right)g\left(x\right)−f\left(x\right)g\left(x\right)+f\left(x\right)g\left(x\right)−f\left(x\right)g\left(x+h\right)}/{h}\right)\left(\lim\_{h\to 0}{1}/{g\left(x\right)g\left(x+h\right)}\right)\\&=\left(\lim\_{h\to 0}{g\left(x\right)\left(f\left(x+h\right)−f\left(x\right)\right)−f\left(x\right)\left(g\left(x+h\right)−g\left(x\right)\right)}/{h}\right)\left(\lim\_{h\to 0}{1}/{g\left(x\right)g\left(x+h\right)}\right)\\&=\left(\left(\lim\_{h\to 0}{g\left(x\right)\left(f\left(x+h\right)−f\left(x\right)\right)}/{h}\right)+\left(\lim\_{h\to 0}{−f\left(x\right)\left(g\left(x+h\right)−g\left(x\right)\right)}/{h}\right)\right)\left(\lim\_{h\to 0}{1}/{g\left(x\right)g\left(x+h\right)}\right)\\&=\left(g\left(x\right)\left(\lim\_{h\to 0}{f\left(x+h\right)−f\left(x\right)}/{h}\right)−f\left(x\right)\left(\lim\_{h\to 0}{g\left(x+h\right)−g\left(x\right)}/{h}\right)\right)\left(\lim\_{h\to 0}{1}/{g\left(x\right)g\left(x+h\right)}\right)\end{matrix}$$

Now, as $h$ tends to $0$, it follows that $g\left(x+h\right)$ tends to $g\left(x\right)$, implying that the final limit tends to $1/\left(g\left(x\right)\right)^{2}$. The other two limits are precisely the definitions of $f′\left(x\right)$ and $g′\left(x\right)$. Therefore, you can write that

$$\begin{matrix}\left(\frac{f}{g}\right)′\left(x\right)&=\left(g\left(x\right)\left(\lim\_{h\to 0}\frac{f\left(x+h\right)−f\left(x\right)}{h}\right)−f\left(x\right)\left(\lim\_{h\to 0}\frac{g\left(x+h\right)−g\left(x\right)}{h}\right)\right)\left(\lim\_{h\to 0}\frac{1}{g\left(x\right)g\left(x+h\right)}\right)\\&=\left(g\left(x\right)f′\left(x\right)−g′\left(x\right)f\left(x\right)\right)⋅\frac{1}{\left(g\left(x\right)\right)^{2}}=\frac{f′\left(x\right)g\left(x\right)−f\left(x\right)g′\left(x\right)}{\left(g\left(x\right)\right)^{2}}\end{matrix}$$

which is the quotient rule.

# Chain rule

*See* [*Guide: The chain rule*](chainrule.qmd) *for more about the chain rule.*

Here is the chain rule, restated with $f\left(x\right)=u\left(x\right)$ and $g\left(x\right)=v\left(x\right)$ for visual ease in the proof that follows.

|  |
| --- |
|  The chain rule |
| Let $f\left(x\right)$ and $g\left(x\right)$ be two differentiable functions. Then the **chain rule** says that$$\left(f∘g\right)′\left(x\right)=\frac{d}{dx}\left(f\left(g\left(x\right)\right)\right)=f′\left(g\left(x\right)\right)⋅g′\left(x\right)$$that is, the derivative of $f\left(x\right)$ composed with $g\left(x\right)$ with respect to $x$ is equal to the product of the derivative of $f$ with respect to $g$ and the derivative of $g$ with respect to $x$.This can also be written as$$\frac{d}{dx}\left(f\left(g\left(x\right)\right)\right)=\frac{df}{dg}⋅\frac{dg}{dx}.$$ |

## Proof of the chain rule

Here’s why the chain rule can be used. The idea is to take the limit definition of $\left(f∘g\right)′\left(x\right)$ and split the limit into the product of the two derivatives $f′\left(g\left(x\right)\right)$ and $g′\left(x\right)$. It requires more thought than the proofs of the product and chain rule, primarily due to the reliance on definitions of differentiation and the fact that it isn’t a creative addition of $0$ that splits the derivative, but a creative multiplication by $1$ instead.

### Alternative definition of derivative

Proving the chain rule requires the restatement of the limit definition of a derivative at a point $a$. Here are the two definitions side by side.

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| --- |
|  Limit definition of the derivative (1) |
| The **derivative of** $f\left(x\right)$ **with respect to** $x$ **at the point** $a$ is defined to be$$f′\left(a\right)=\lim\_{h\to 0}\frac{f\left(a+h\right)−f\left(a\right)}{h}.$$ |

|  |
| --- |
|  Limit definition of the derivative (2) |
| The **derivative of** $f\left(x\right)$ **with respect to** $x$ **at the point** $a$ is defined to be$$f′\left(a\right)=\lim\_{x\to a}\frac{f\left(x\right)−f\left(a\right)}{x−a}.$$ |

(See [Guide: Introduction to differentiability] for more.) To see that these are equal, start with definition (1). Here, $h$ is the variable as the limit depends on $h$. Now, rescale the limit by setting $h=x−a$ (see [Guide: Properties of limits] for more). This gives

$$\begin{matrix}f′\left(a\right)&=\lim\_{h\to 0}\frac{f\left(a+h\right)−f\left(a\right)}{h}\\&=\lim\_{x−a\to 0}\frac{f\left(a+\left(x−a\right)\right)−f\left(a\right)}{x−a}.\end{matrix}$$

As $x−a\rightarrow 0$, it follows that $x\rightarrow a$; in addition, $a+\left(x−a\right)=x$. So the limit becomes

$$f′\left(a\right)=\lim\_{x−a\to 0}\frac{f\left(a+\left(x−a\right)\right)−f\left(a\right)}{x−a}=\lim\_{x\to a}\frac{f\left(x\right)−f\left(a\right)}{x−a}$$

so the definitions are the same at a point. Since a function $f$ is differentiable on an interval $I$ of real numbers if and only if $f′\left(a\right)$ exists for all $a$ in $I$, it follows that you can use this definition for a differentiable function.

### Intuition

The idea is to start with the second limit definition of the derivative above and put the function $\left(f∘g\right)\left(x\right)=f\left(g\left(x\right)\right)$ into the definition to get:

$$\left(f∘g\right)′\left(a\right)=\lim\_{x\to a}\frac{f\left(g\left(x\right)\right)−f\left(g\left(a\right)\right)}{x−a}.$$

Now, you would want to generate the derivative of $f$ with respect to $g\left(x\right)$ at $a$ and the derivative of $g$ with respect to $x$ at $a$. To do this, you can notice that the $x−a$ is already there for the derivative of $g$. You can multiply top and bottom of the fraction by $g\left(x\right)−g\left(a\right)$. Since $\frac{\left(g\left(x\right)−g\left(a\right)\right)}{\left(g\left(x\right)−g\left(a\right)\right)}=1$, this does not change the value of the limit. This gives

$$\left(f∘g\right)′\left(a\right)=\lim\_{x\to a}\left(\frac{f\left(g\left(x\right)\right)−f\left(g\left(a\right)\right)}{x−a}⋅\frac{\left(g\left(x\right)−g\left(a\right)\right)}{\left(g\left(x\right)−g\left(a\right)\right)}\right).$$

You can now pull this limit apart to attempt to make the two definitions of $f′\left(g\left(a\right)\right)$ and $g′\left(a\right)$. Using the properties of limits to do this gives

$$\begin{matrix}\left(f∘g\right)′\left(a\right)&=\lim\_{x\to a}\left(\frac{f\left(g\left(x\right)\right)−f\left(g\left(a\right)\right)}{x−a}⋅\frac{\left(g\left(x\right)−g\left(a\right)\right)}{\left(g\left(x\right)−g\left(a\right)\right)}\right)\\&=\lim\_{x\to a}\frac{f\left(g\left(x\right)\right)−f\left(g\left(a\right)\right)}{g\left(x\right)−g\left(a\right)}⋅\lim\_{x\to a}\frac{\left(g\left(x\right)−g\left(a\right)\right)}{x−a}.\end{matrix}$$

The second of these terms is $g′\left(a\right)$, which is what you want. The first of these terms **would** be the definition of $f′\left(g\left(a\right)\right)$… if the limit was $g\left(x\right)\rightarrow g\left(a\right)$ rather than $x\rightarrow a$. Here is the problem, because you **cannot guarantee** the behaviour of $g\left(x\right)−g\left(a\right)$ as $x$ gets closer to $a$; it could be that $g\left(x\right)−g\left(a\right)=0$, which is a big problem. In fact, it could be that as $x$ gets closer to $a$, then $g\left(x\right)−g\left(a\right)$ could be $0$ in infinitely many different places. This needs to be rectified.

### Overcoming the technicality

The idea is to ‘fill in’ the places where $g\left(x\right)−g\left(a\right)=0$, by defining the value of the function $\frac{f\left(g\left(x\right)\right)−f\left(g\left(a\right)\right)}{g\left(x\right)−g\left(a\right)}$ at these points. You can define the function

$$ϕ\left(y\right)=\left\{\begin{matrix}\frac{f\left(y\right)−f\left(g\left(a\right)\right)}{y−g\left(a\right)}& if y\ne g\left(a\right)\\f′\left(g\left(a\right)\right)& if y=g\left(a\right)\end{matrix}\right.$$

You can notice here that $f′\left(g\left(a\right)\right)$ is already defined as $f$ is a differentiable function, meaning that $f′\left(y\right)$ exists for all $y$.

Now, consider the expression

$$ϕ\left(g\left(x\right)\right)⋅\frac{\left(g\left(x\right)−g\left(a\right)\right)}{x−a}.$$

The idea is to prove that

$$\frac{f\left(g\left(x\right)\right)−f\left(g\left(a\right)\right)}{x−a}=ϕ\left(g\left(x\right)\right)⋅\frac{\left(g\left(x\right)−g\left(a\right)\right)}{x−a}$$

for all $x$. This way, you can evaluate the limit of the right hand side instead of the left hand side. However, this does depend on whether or not $g\left(x\right)=g\left(a\right)$.

* If $g\left(x\right)\ne g\left(a\right)$, then $g\left(x\right)−g\left(a\right)\ne 0$. You can use the first part of the definition to say that

$$ϕ\left(g\left(x\right)\right)⋅\frac{\left(g\left(x\right)−g\left(a\right)\right)}{x−a}=\frac{f\left(g\left(x\right)\right)−f\left(g\left(a\right)\right)}{g\left(x\right)−g\left(a\right)}⋅\frac{\left(g\left(x\right)−g\left(a\right)\right)}{x−a}$$

* Since $g\left(x\right)−g\left(a\right)\ne 0$, you can cancel these to get

$$ϕ\left(g\left(x\right)\right)⋅\frac{\left(g\left(x\right)−g\left(a\right)\right)}{x−a}=\frac{f\left(g\left(x\right)\right)−f\left(g\left(a\right)\right)}{x−a}.$$

* If $g\left(x\right)=g\left(a\right)$ then $g\left(x\right)−g\left(a\right)=0$ and also $f\left(g\left(x\right)\right)−f\left(g\left(a\right)\right)=0$. This implies that

$$ϕ\left(g\left(x\right)\right)⋅\frac{\left(g\left(x\right)−g\left(a\right)\right)}{x−a}=ϕ\left(g\left(x\right)\right)⋅0=0.$$

So they really are equal. Using this expression, together with the properties of limits gives

$$\begin{matrix}\left(f∘g\right)′\left(a\right)&=\left(\frac{f\left(g\left(x\right)\right)−f\left(g\left(a\right)\right)}{x−a}\right)\\&=\lim\_{x\to a}ϕ\left(g\left(x\right)\right)⋅\lim\_{x\to a}\frac{g\left(x\right)−g\left(a\right)}{x−a}\end{matrix}$$

The idea is then to prove that these two limits exist; as then $\left(f∘g\right)′\left(a\right)$ would exist. The second of these limits is precisely the definition of $g′\left(a\right)$, so let’s focus on the limit of $ϕ\left(g\left(x\right)\right)$ as $x$ tends to $a$. If this function $ϕ∘g$ is continuous at $a$ (see [Guide: Introduction to continuity]) then this limit exists and is equal to $ϕ\left(g\left(a\right)\right)$. The function $ϕ$ is defined whenever $f$ is. Since $f$ is differentiable, then it is continuous at every point, including $g\left(a\right)$; therefore, $ϕ$ is continuous at $g\left(a\right)$. Since $g$ is differentiable at $a$, then $g$ is continuous at $a$. Therefore, by properties of continuous functions (see [Guide: Introduction to continuity]), $ϕ∘g$ is continuous at $a$. It follows that

$$\lim\_{x\to a}ϕ\left(g\left(x\right)\right)=ϕ\left(g\left(a\right)\right)=f′\left(g\left(a\right)\right)$$

by definition and so

$$\begin{matrix}\left(f∘g\right)′\left(a\right)&=\lim\_{x\to a}ϕ\left(g\left(x\right)\right)⋅\lim\_{x\to a}\frac{g\left(x\right)−g\left(a\right)}{x−a}\\&=f′\left(g\left(a\right)\right)⋅g′\left(a\right)\end{matrix}$$

and this is the chain rule!

# Further reading

[Click this link to go back to Guide: Introduction to differentiation and the derivative.](../studyguides/introtodifferentiation.qmd)

[Click this link to go back to Guide: The product rule.](../studyguides/productrule.qmd)

[Click this link to go back to Guide: The quotient rule.](../studyguides/quotientrule.qmd)

[Click this link to go back to Guide: The chain rule](../studyguides/chainrule.qmd)

[For questions on differentiation and the derivative, please go to Questions: Introduction to differentation and the derivative.](../questions/qs-introtodifferentiation.qmd)

## Version history

v1.0: initial version created in 05/25 by tdhc.

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