Proof: the quadratic formula

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Summary

An explanation as to why the quadratic formula is true.

*Before reading this proof sheet, it is recommended that you read* [*Guide: Completing the square*](../studyguides/completingthesquare.qmd)*.*

# Proof of the quadratic formula

Remember from [Guide: Using the quadratic formula](../studyguides/quadraticformula.qmd) that the **quadratic formula** is used to find roots of any quadratic equation:

|  |
| --- |
|  The quadratic formula |
| Let $ax^{2}+bx+c=0$ be a quadratic equation (where $a\ne 0$). The roots to this quadratic equation are given by$$x=\frac{−b\pm \sqrt{b^{2}−4ac}}{2a}$$where one of the roots is given by the term $\left(−b+\sqrt{b^{2}−4ac}\right)/2a$ and the other given by the term $\left(−b−\sqrt{b^{2}−4ac}\right)/2a$. |

In order to prove that these really are the solutions to the quadratic, you can **complete the square** on $ax^{2}+bx+c$ using the fact that $a\ne 0$. See [Guide: Completing the square](../studyguides/completingthesquare.qmd) for why this works.

## Proof of the quadratic formula

First of all, as $a\ne 0$ you can divide $ax^{2}+bx+c=0$ through by $a$ to get

$$x^{2}+\frac{b}{a}x+\frac{c}{a}=0$$

Taking the $c/a$ term over to the other side gives

$$x^{2}+\frac{b}{a}x=−\frac{c}{a}$$

Completing the square (see [Guide: Completing the square](../studyguides/completingthesquare.qmd)) gives

$$\left(x+\frac{b}{2a}\right)^{2}−\frac{b^{2}}{4a^{2}}=−\frac{c}{a}$$

You can rearrange to get

$$\left(x+\frac{b}{2a}\right)^{2}=\frac{b^{2}}{4a^{2}}−\frac{c}{a}=\frac{b^{2}−4ac}{4a^{2}}$$

Now the result is starting to come together. Taking square roots of both sides (not forgetting that it could be positive or negative) gives

$$x+\frac{b}{2a}=\pm \frac{\sqrt{b^{2}−4ac}}{2a}$$

and rearranging gives

$$x=\frac{−b\pm \sqrt{b^{2}−4ac}}{2a}$$

as required.

# Further reading

[Guide: Using the quadratic formula](../studyguides/quadraticformula.qmd)

[Questions: Using the quadratic formula](../questions/qs-quadraticformula.qmd)

## Version history and licensing

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