Proof: Law of total probability and Bayes’ theorem

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Summary

This proof sheet demonstrates that the law of total probability and Bayes’ theorem are true.

*Before reading this proof sheet, it is recommended that you read* [*Guide: Conditional probability*](../studyguides/conditionalprobability.qmd) *and* [*Guide: Law of total probability and Bayes’ theorem*](../studyguides/bayestheorem.qmd)*.*

# Proof of the law of total probability

First of all, here is a restatement of the law of total probability from [Guide: Law of total probability and Bayes’ theorem](../studyguides/bayestheorem.qmd):

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|  Definition of the law of total probability |
| Suppose an event $B$ depends on several possible scenarios. These scenarios can be described by events $A\_{1},A\_{2},…,A\_{n}$, that are:* **Mutually exclusive**: they cannot occur at the same time, and
* **Exhaustive**: one of them must always occur.

Then, the **law of total probability** states that the probability of event $B$ is:$$P\left(B\right)=\sum\_{i=1}^{n}P\left(A\_{i}\right)P\left(B∣A\_{i}\right)$$ |

The proof of the law of total probability comes directly from the definition of conditional probability given in [Guide: Conditional probability](../studyguides/conditionalprobability.qmd):

$$P\left(B∣A\_{i}\right)=\frac{P\left(B∩A\_{i}\right)}{P\left(A\_{i}\right)}$$

Multiplying by $P\left(A\_{i}\right)$ gives the multiplication rule (again from [Guide: Conditional probability](../studyguides/conditionalprobability.qmd)) :

$$P\left(B∩A\_{i}\right)=P\left(B∣A\_{i}\right)P\left(A\_{i}\right)$$

As scenarios $A\_{1},A\_{2},…,A\_{n}$ are mutually exclusive (so $A\_{i}∩A\_{j}=∅$ for all $1\leq i\ne j\leq n$) and exhaustive ($⋃\_{1\leq i\leq n}A\_{i}=B$), it follows from results in set theory (see [Guide: Operations on sets]) that:

$$P\left(B\right)=P\left(B∩A\_{1}\right)+P\left(B∩A\_{2}\right)+…+P\left(B∩A\_{n}\right)=\sum\_{i=1}^{n}P\left(B∩A\_{i}\right)$$

Substituting the above expressions gives:

$$P\left(B\right)=P\left(B∣A\_{1}\right)P\left(A\_{1}\right)+P\left(B∣A\_{2}\right)P\left(A\_{2}\right)+…+P\left(B∣A\_{n}\right)P\left(A\_{n}\right)=\sum\_{i=1}^{n}P\left(B∣A\_{i}\right)P\left(A\_{i}\right)$$

Which results in the law of total probability:

$$P\left(B\right)=\sum\_{i=1}^{n}P\left(A\_{i}\right)P\left(B∣A\_{i}\right)$$

# Proof of Bayes’ theorem

Here is the statement of Bayes’ theorem from [Guide: Law of total probability and Bayes’ theorem](../studyguides/bayestheorem.qmd):

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| --- |
|  Statement of Bayes’ Theorem |
| If $A$ and $B$ are events with $P\left(B\right)>0$, then Bayes’ Theorem states:$$P\left(A∣B\right)=\frac{P\left(B∣A\right)P\left(A\right)}{P\left(B\right)}.$$where:* $P\left(A∣B\right)$ is the probability of $A$ given $B$,
* $P\left(B∣A\right)$ is the probability of $B$ given $A$,
* $P\left(A\right)$ and $P\left(B\right)$ are the individual probabilities of $A$ and $B$, respectively.
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Bayes’ Theorem is derived directly from the definition of conditional probability: see [Guide: Conditional probability](../studyguides/conditionalprobability.qmd). Start with the conditional probabilities of two events $A$ and $B$:

$$\left(1\right) P\left(A∣B\right)=\frac{P\left(A∩B\right)}{P\left(B\right)}  and  \left(2\right) P\left(B∣A\right)=\frac{P\left(A∩B\right)}{P\left(A\right)}$$

You can rearrange $\left(2\right)$ by multiplying both sides by $P\left(A\right)$, giving the multiplication rule:

$$P\left(A∩B\right)=P\left(B∣A\right)P\left(A\right)$$

Substitute this result into equation $\left(1\right)$ to get:

$$P\left(A∣B\right)=\frac{P\left(B∣A\right)P\left(A\right)}{P\left(B\right)}$$

This gives Bayes’ Theorem, a way to reverse conditional probabilities when direct calculation is difficult.

# Further reading

[Click this link to go back to Guide: Law of total probability and Bayes’ theorem.](../studyguides/bayestheorem.qmd)

## Version history

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