# Making it count: an introduction to the theory of functions

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**Summary**

In the third of four linked presentations, this session takes the material covered in the previous two sessions and looks at it from a more general perspective by introducing the theory of functions. Along the way, you'll learn why and how you use functions in everyday life, how every sequence of numbers is a function in disguise, and why pigeons are so important in mathematics.

## The function junction

**! Try it yourself 1**

How many elements are in this set?

$$X = \{a, 1, 2, 3, b, 5, 193, 14555, c, p, 23, L, Q, W, A\}$$

**i Answer to try it yourself 1**

There are $14$, but that's not important. Read on...

The important thing was **how** you counted it. If you counted out loud, you probably said $1, 2, 3, 4, \ldots$ all the way up to $14$, pointing at each element in turn. This is how you are first taught to count things.

In fact, what you are doing is taking an input from $X$ and assigning it an output from the set of all positive whole numbers. This linking of an input to an output is the idea of a mathematical **function**.

**i Function, domain, range**

Let $X, Y$ be two sets.

A **function** $f$ from the set $X$ to the set $Y$ is an assignment that maps each element of $X$ to **exactly one** output in $Y$. It is written as $f : X \to Y$, and the notation $f(x) = y$

means '$f$ maps $x \in X$ to $y \in Y$.
Here, the set $X$ is called the **domain** of $f$ and the set $Y$ is called the **codomain** of $f$.
Two functions $f$ and $g$ are equal if and only if they have the same domain $X$, codomain $Y$, and $f(x) = g(x)$ for all $x \in X$.

The important thing is that for some $x \in X$, there is exactly one $y \in Y$ such that $f(x) = y$ - so **one** element of $X$ can't map to **many** elements of $Y$. The converse is not true; **many** elements of $X$ are allowed to **one** element of $Y$.

Here are some examples of functions.

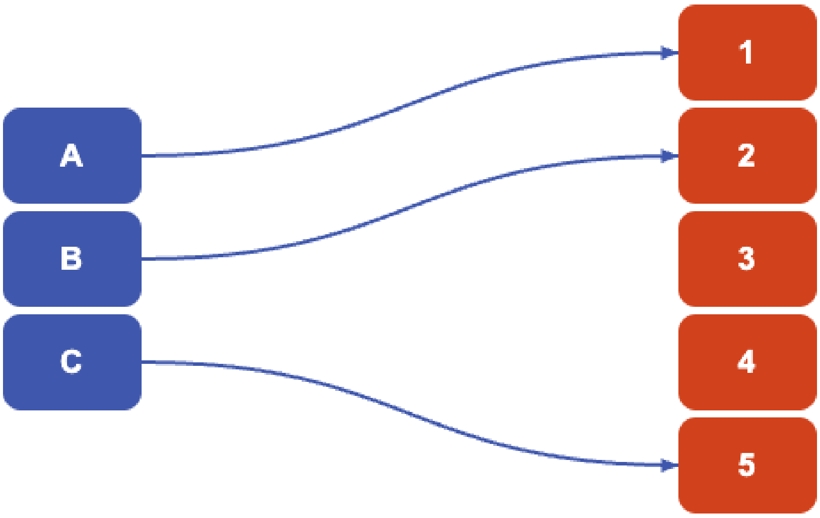


Figure 1: This is an example of a function. You can notice each element of the domain $\{A, B, C\}$ on the left is mapped to exactly one element in the codomain $\{1, 2, 3, 4, 5\}$ on the right. $A$ is mapped to 1, $B$ is mapped to 2, and $C$ is mapped to 5.

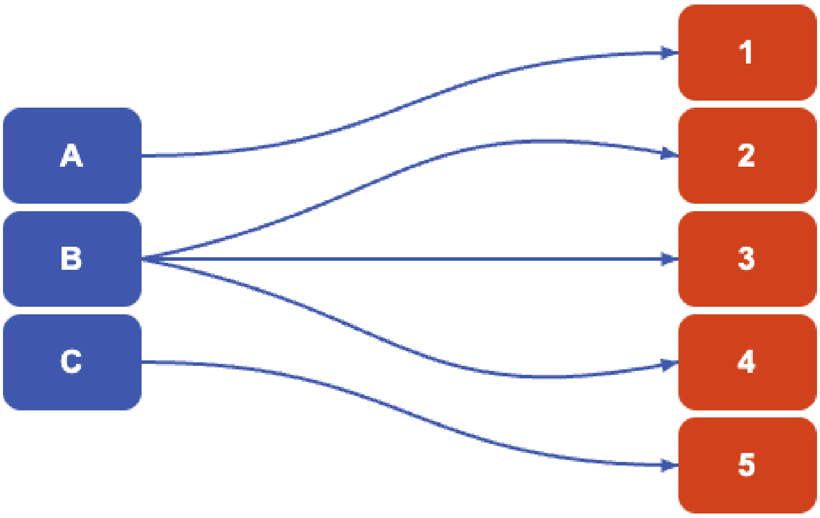


Figure 2: This is an example of something that isn't a function, because it shows 'one-to-many' behaviour. The element $B$ can only be mapped to one element in a function, but here it is mapped to three elements $2, 3, 4$.

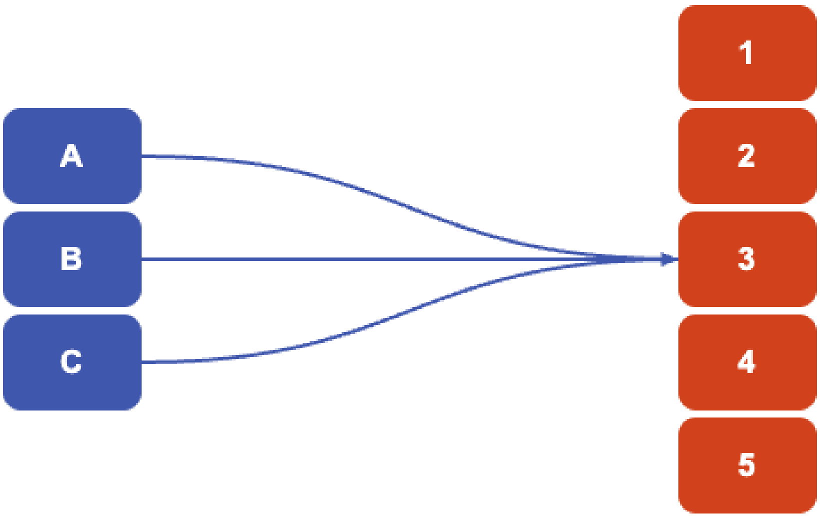


Figure 3: This is an example of a function showing 'many-to-one' behaviour. You can notice each element of the domain $\{A, B, C\}$ on the left is mapped to exactly one element in the codomain $\{1, 2, 3, 4, 5\}$ on the right. All elements of the domain are mapped to $3$.

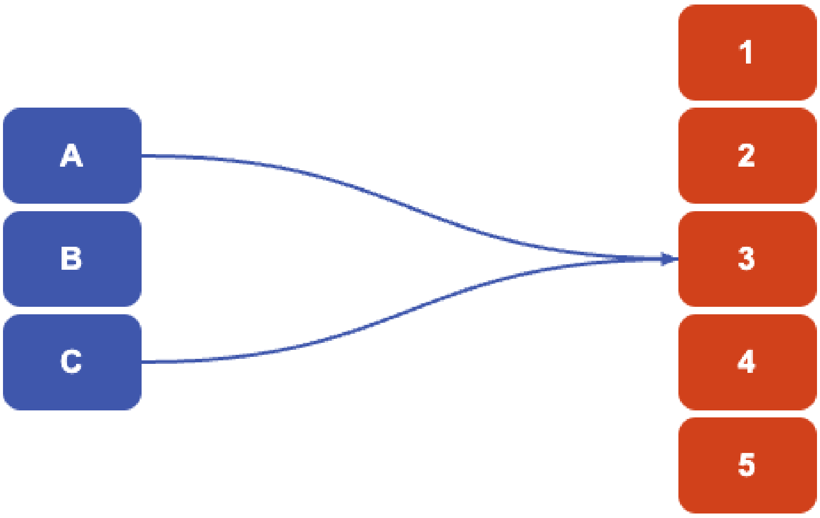


Figure 4: This is an example of something that isn't a function. Not every element of the domain $\{A, B, C\}$ is mapped to an element of the codomain. To turn this into a function, you will need to decide where $B$ is mapped to.

## Sequences

In mathematics, the reason why functions are incredibly powerful is because they allow for **comparison** between sets. Any time you want to compare two sets - a function is almost always used. Any time you want to look at growth, or rates of change, or to decide behaviour in the future: you will use a function.

One of the ways that functions are studied in mathematics is the idea of a **sequence**. You saw all about sequences in Exploration: The hidden sequences of Pascal's triangle; how they can be either finite or infinite, how they could have formulas or not.

In fact, every sequence is a function in disguise. To help illustrate this, here is some new mathematical notation; this is a common thing to do in mathematics to help explain certain concepts.

**Special notation for sets**

Write $[n]$ to be the set $\{1, 2, 3, \ldots, n\}$ of all positive whole numbers from $1$ up to $n$ included. So $[4] = \{1, 2, 3, 4\}$ and $[k] = \{1, 2, 3, \ldots, k\}$. You can notice that the number of elements of $[n]$ is $n$
For the (infinite) set of all positive whole numbers, write

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

as the set of **natural numbers**. You can write

$$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$$

to be the set of **natural numbers with zero**.
For the (infinite) set of all possible numbers - whole, fractions and decimals, positive and negative - write $\mathbb{R}$ to be the set of these numbers (called **real numbers**).

This makes it a lot easier to phrase the following important consideration:

**Sequences and functions**

(a) Any finite sequence of length $n$ of symbols from the set $[k]$ can be viewed as a function $f : [n] \to [k]$.

(b) In the reverse direction, any function $f : [n] \to [k]$ is a finite sequence of length $n$ of symbols from the set $[k]$

(c) Any finite sequence of numbers of length $n$ can be viewed as a function $f : [n] \to \mathbb{R}$.

(d) Any infinite sequence of numbers can be viewed as a function $f : \mathbb{N} \to \mathbb{R}$, and any function $f : \mathbb{N} \to \mathbb{R}$ can be written as an infinite sequence.

**Try it yourself 2**

Why are (a) and (b) true? As an initial pointer, think about where the set $[n]$ will appear in your consideration.

**Answer to try it yourself 2**

For (a), suppose that you have a sequence of length $n$ of elements from $[k]$; you could say that this is

$$a_1, a_2, \dots, a_n.$$

You can see that for every number $i$ between $1$ and $n$, there is an element $a_i$ of $[k]$ at position $i$ in the list. You can then define a function $f : [n] \to [k]$ by the assignment

$$f(i) = a_i$$

which sends $i \in [n]$ to $a_i$. This then determines the sequence $a_1, a_2, \dots, a_n$.
For (b), suppose you have a function $f : [n] \to [k]$. You can now view this as a sequence

of length $n$ in $[k]$ by writing out all of the outputs in order:

$$f(1), f(2), \dots, f(n)$$

which defines a finite sequence of length $n$ with elements in $[k]$.

Parts (c) and (d) follow from similar reasoning.

This means that finite/infinite sequences can be studied through functions.

**Examples of sequences as functions**

If a sequence has a **closed form formula**, then it can be represented concisely as a function. For instance, the sequence of cube numbers can be written as $f_1 : \mathbb{N} \to \mathbb{N}$ with $f(n) = n^3$, or the powers of two as $f_2 : \mathbb{N}_0 \to \mathbb{N}$ with $f_2(n) = 2^n$. Even constant sequences can be written as functions: write $f_3 : \mathbb{N} \to \{153\}$ to be $f_3(n) = 153$ for all $n \in \mathbb{N}$.
The binomial coefficients can also be written as a function, but the domain needs to be all **pairs** of all whole numbers with $0$. Write

$$\mathbb{N}_0 \times \mathbb{N}_0 \ = \{(n,k) \ : \ n,k \in \mathbb{N}_0\}.$$

Then the binomial coefficient can be written as a function $b : \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}$ with

$$b(n,k) = \binom{n}{k}.$$

Here's an excellent result that follows from this.

**Counting number of functions**

The total number of possible functions $f : [n] \to [k]$ is $k^n$.

**Proof of the formula of the number of functions**

You are given that $[n] = \{1, 2, 3, \dots, n\}$, and there is a function $f : [n] \to [k]$. Write $f$ as a finite sequence of length $n$ with elements from $k$ to get

$$f(1), f(2), f(3), \dots, f(n)$$

So, you can ask yourself - what is $f(1)$? Well, you don't know, as you aren't given the information. However, there are $k$ many **choices** for the value of $f(1)$, because it has to be exactly one element of the set $[k] = \{1, 2, 3, \dots, k\}$. Since it can be any of them, this gives $k$ choices.

Next, you can ask - what is $f(2)$? Again, there are $k$ many choices. You can notice here that the choice of the value of $f(2)$ is completely independent of the choice for the value of $f(1)$. Similarly, there are $k$ many choices for the value of $f(3)$ and this choice is independent of the values of $f(1)$ and $f(2)$.

You can carry on in this way for all the elements of $[n]$. With $k$ choices for each element, and all of the choices being independent of each other, this leads to

$$\underbrace{k}_{\text{what is } f(1)?} \cdot \underbrace{k}_{\text{what is } f(2)?} \cdot \underbrace{k}_{\text{what is } f(3)?} \cdot \dots \cdot \underbrace{k}_{\text{what is } f(n)?} = k^n$$

many choices for the sequence $f(1), f(2), f(3), \dots, f(k)$. Therefore, there are $k^n$ possible functions $f : [n] \to [k]$.

**Try it yourself 3**

Does this seem familiar?

**Note**

This is an almost a word-for-word recreation of the proof in Exploration: How to win at cards - combinations and permutations that there were $2^n$ many subsets of a set $\{1, 2, 3, \dots, n\}$!

So what's happening here? Well, the idea of deciding whether or not each element of $[n] = \{1, 2, 3, \dots, n\}$ is in some subset of $[n]$ can be modelled as a function $f : [n] \to [2]$, where $f(i) = 1$ says that $i$ is not in the subset, and $f(i) = 2$ says that $i$ is in the subset. Therefore, every function $f : [n] \to [2]$ corresponds to a subset of $[n]$, and every subset of $[n]$ corresponds to a function $f : [n] \to [2]$. This function is called an **indicator function** for the subset of $[n]$.

## Injective, surjective, bijective

Now, much like numbers can be odd, even, prime, not prime, and so on, functions have different properties as well. To do this, you'll need to examine **exactly** where the function sends its elements.

**Image of a function**

The set

$$f(X) = \{f(x) \text{ where } x \in X\} \subseteq Y$$

is called the **image** of $f$. It follows from this that the image of $f$ is a subset of the codomain $Y$ of $f$.

Here are some examples of images of functions.

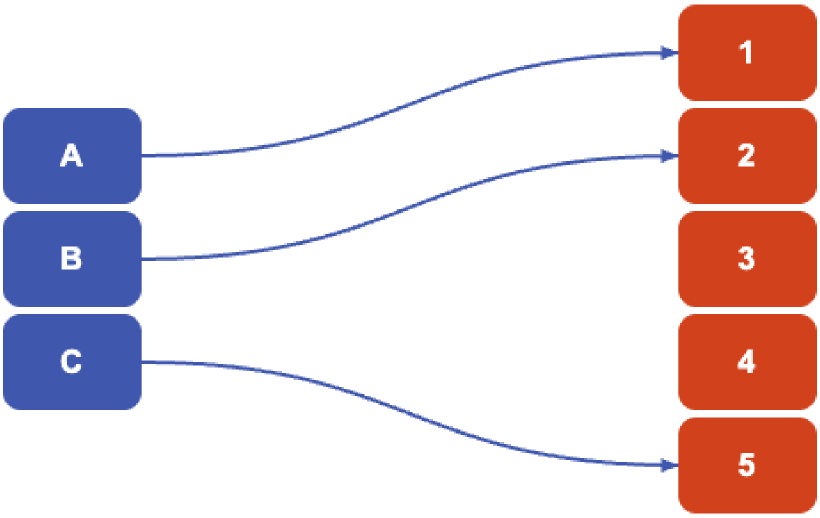


Figure 5: In this function $A$ is mapped to $1$, $B$ is mapped to $2$, and $C$ is mapped to $5$, so the image of this function is $\{1, 2, 5\}$.

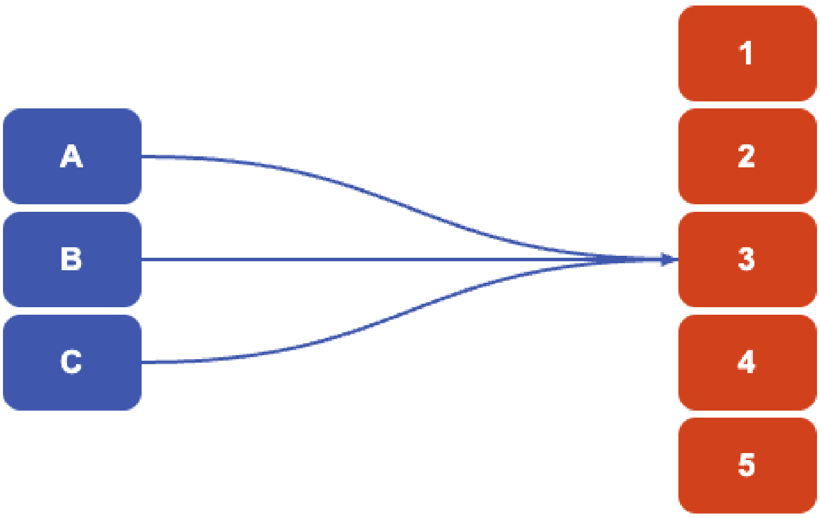


Figure 6: As everything in the domain $\{A, B, C\}$ is mapped to $3$, the image of this function is the set $\{3\}$, which is a subset of the codomain $\{1, 2, 3, 4, 5\}$.

How functions map into the image is important, and how much of the codomain is covered in the image is important. Together, these properties underpin the very idea of counting.

**Injective, surjective, and bijective**

Suppose that $f : X \to Y$ is a function from $X$ to $Y$.

(a) Say that $f$ is **injective** (or **one-to-one**, or even **one-one**) if $x_1 \neq x_2$ means that $f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in X$. Equivalently, $f$ is injective if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$ for all $x_1, x_2 \in X$.

(b) Say that $f$ is **surjective** (or **onto**) if for all $y \in Y$ there exists $x \in X$ such that $f(x) = y$. Equivalently, $f$ is surjective if the image $f(X)$ of $f$ is equal to the codomain $Y$ of $f$.

(c) Say that $f$ is **bijective** if $f$ is both injective and surjective.

Intuitively, you could say that

- $f$ is injective if different elements in $A$ always map to different elements in $B$.
- $f$ is surjective if every element of $B$ has an element of $A$ that maps to it.
- $f$ is bijective if every element of $B$ has a **unique** element of $A$ that maps to it.

It's important that in mathematics, you have both the **precise definition** and the **intuition behind it** - it's like the spirit and letter of the law, they have to line up in order to be effective.

**Try it yourself 4**

Using the interactive function investigator above, find examples of injective, surjective and bijective functions, and write down their images. What do you notice about the sizes of the sets involved?

**Answer to try it yourself 4**

Here's an example of an injective function $f : \{A, B, C\} \to \{1, 2, 3, 4, 5\}$:

$$f(A) = 1, \quad f(B) = 2, \quad f(C) = 5$$

with image $\{1, 2, 5\}$. You can notice that each of the domain elements $\{A, B, C\}$ is sent to a different element of the codomain, demonstrating that the function is injective.

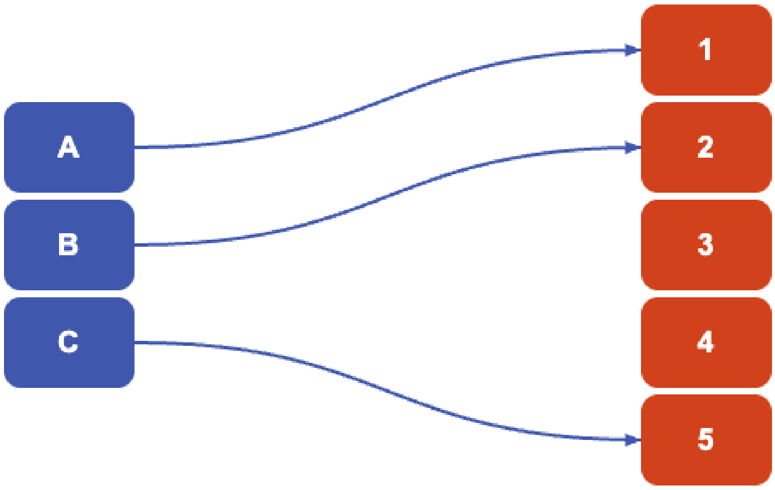


Figure 7: A picture of the injective function in the answer to Try it yourself 4.

Here's an example of a surjective function $g : \{A, B, C, D, E\} \to \{1, 2, 3\}$:

$$g(A) = 1, \quad g(B) = 2, \quad g(C) = 3, \quad g(D) = 1, \quad g(E) = 2$$

with image $\{1, 2, 3\}$. Since the image $\{1, 2, 3\}$ is the same as the codomain, it follows that this function is surjective.

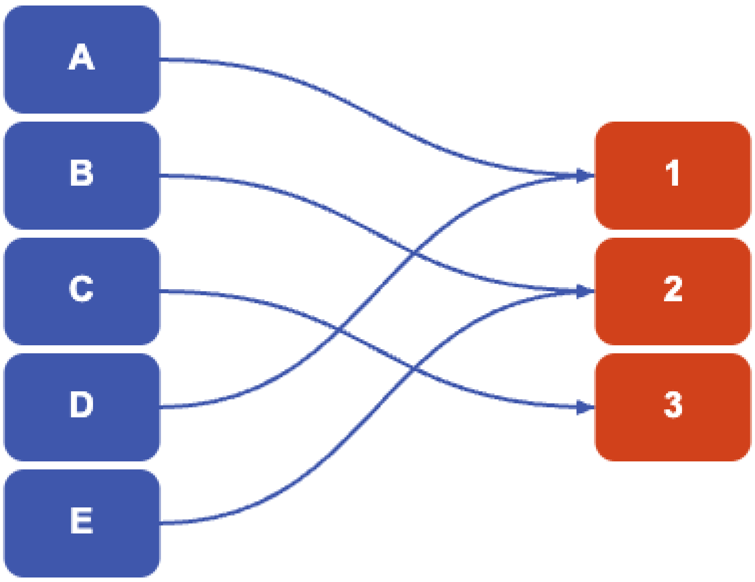


Figure 8: A picture of the surjective function in the answer to Try it yourself 4.

Here's an example of a bijective function $h : \{A, B, C\} \to \{1, 2, 3\}$:

$$h(A) = 1, \quad h(B) = 2, \quad h(C) = 3$$

with image $\{1, 2, 3\}$. You can notice that each of the domain elements $\{A, B, C\}$ is sent to a different element of the codomain, demonstrating that the function is injective - in addition, the image $\{1, 2, 3\}$ is the same as the codomain, so the function is also surjective. This means that $h$ is bijective.

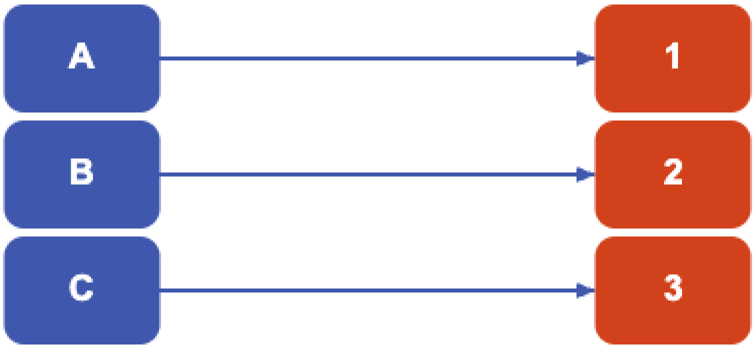


Figure 9: A picture of the bijective function in the answer to Try it yourself 4.

What about the functions before?

**Previous examples**

- The function $f_1 : \mathbb{N} \to \mathbb{N}$ with $f_1(n) = n^3$ is injective but not surjective. For injective, you need to take two potential equal ouputs and show they come from the same input. So if $f_1(x) = f_1(y)$ then $x^3 = y^3$, and so $x = y$ - therefore, $f_1$ is injective. To show that $f_1$ is not surjective, it's enough to show that there is an element of the codomain that isn't in the image; since there is no natural number that cubes to make 2, it follows that 2 is not in the image. So $f_1$ is not surjective.
- Similarly, $f_2 : \mathbb{N}_0 \to \mathbb{N}$ with $f_2(n) = 2^n$ is injective but not surjective.
- The function $f_3 : \mathbb{N} \to \{153\}$ with $f_3(n) = 153$ for all $n \in \mathbb{N}$ is surjective but not injective. Here, the image of $f_3$ is $\{153\}$ which is the same as the codomain, so $f_3$ is surjective. To show that it's not injective, you can notice that $f_3$ sends both 1 and 2 to 153 and since $1 \neq 2$, different elements aren't always mapped to different elements and the function is not injective.

- The binomial coefficient written as a function $b : \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}$ with

$$b(n,k) = \binom{n}{k}$$

is surjective but not injective. For surjective, suppose that $m$ is in $\mathbb{N}$; you need to find a pair $(n,k)$ such that $b(n,k) = m$. Since $\binom{m}{1} = m$ for all $m$ in $\mathbb{N}$, it follows that taking $n = m$ and $k = 1$ shows that every $m \in \mathbb{N}$ is in the image of $b$. So $b$ is surjective. However, $b$ is not injective as

$$b(3,1) = \binom{3}{1} = 3 = \binom{3}{2} = b(3,2)$$

but the inputs $(3,1)$ and $(3,2)$ are not the same.

You can also notice that

- if a function is surjective, then the cardinality of the domain $X$ is greater than or equal to the cardinality of the codomain $Y$, so $|X| \geq |Y|$.
- if a function is injective, then the cardinality of the codomain $Y$ is greater than or equal to the cardinality of the domain $Y$, so $|Y| \geq |X|$.
- if a function is bijective, then the cardinality of the domain $X$ is equal to the cardinality of the domain $Y$, so $|X| = |Y|$.

These observations can be reversed using a mathematical principle called the **contrapositive**. This allows you to change the logical perspective of a statement 'if $A$ then $B$' to 'if not $B$ then not $A$'.

Doing this to the observations above give the following statements about any function $f : X \to Y$:

- if the cardinality of the domain $X$ is **less than** the cardinality of the codomain $Y$, then $f$ is never surjective.
- if the cardinality of the codomain $Y$ is **less than** the cardinality of the domain $X$, then $f$ is never injective.
- if the cardinalities of the domain $X$ and the codomain $Y$ are **not equal**, then $f$ is never bijective.

## Pigeons

The second of these statements is equivalent to an important principle in combinatorics.

**The pigeonhole principle**

Suppose that $n > m$. If you have $n$ items and $m$ boxes in which to put them, then at least one box contains more than one item.

**Proof of the pigeonhole principle**

Give numbers to your list of items $\{1, 2, 3, \dots, n\}$. Put item $1$ in box $1$, item $2$ in box $2$, and so on until you put item $m$ in box $m$. Since $n > m$, it follows that $n - m \geq 1$ and so you have at least one item to put in a box. It follows that at least one box contains more than one item.

**Scenario 1**

Cantor's Confectionery is one of the best places to work in the fictional world. It has the habit of throwing a birthday party for every one of its employees on their birthday, which is truly exceptional for morale. They have appointed you, its HR representative, to try and investigate cost-cutting measures for this expensive endeavour. The best way to achieve this is to have a shared party between two or more employees.
What is the number $n$ of employees required to have

(a) a $50\%$ chance that two employees share the same birthday?

(b) a $100\%$ chance that two employees share the same birthday?

The first of these is probability, and relies on techniques seen in Explorations: How to win at cards - combinations and permutations. The second uses the pigeonhole principle!

**Answer to Scenario 1**

For convenience, assume that there are $365$ days in the year, as there are in most years.

(a) Here, it's easier to work out the probability of this **not** happening - you can then subtract this from $1$ to work out the probability of this happening.

Write a list of your employees. Suppose that the first employee has a birthday on any day of the year. The chances that a second employee has a different birthday than the first is $364/365$. The chances that a third employee has a different birthday than the first two is $363/365$. You can continue in this way to show that the probability that $n$

employees all have different birthdays is

$$\begin{aligned}\mathbb{P}(\text{different birthdays}) &= \frac{365}{365}\cdot\frac{364}{365}\cdot\frac{363}{365}\cdot\ldots\cdot\frac{(365-n)}{365}\\ &= \frac{(365)_n}{365^n}\end{aligned}$$

where $(365)_n$ is the **falling factorial** from Explorations: How to win at cards - combinations and permutations.

As it turns out, the number $n$ that tips this number below $50\%$ - and so the chances that two employees share a birthday is **above** $50\%$ - is $n = 23$. So for a company with $23$ employees, there is a greater than $50\%$ chance that two employees share a birthday!

(b) Here, the **worst-case scenario** is that everybody has a different birthday. Since there are $365$ possible birthdays, the worst case scenario is that the company has $365$ employees, each with different birthdays.

So if there are $366$ employees in Cantor's Confectionery, then it's guaranteed that at least two people share a single birthday. So the answer is $n = 366$.

The pigeonhole principle also guarantees that in Scotland, at least two people have the same numbers of hairs on their head!

## Counting blessings

So what does all of this have to do with counting?

If you count the set in Try it yourself 1 by pointing at each element and saying each number, you created a bijection between the set and the set of number $\{1, 2, \ldots, 14\}$. By putting sets in bijections with other sets, or not, you can directly compare them to each other by their sizes. This is infact the definition of counting:

**Definition of cardinality**

Let $X$ be a set, and $[n] = \{1, 2, 3, \ldots, n\}$. If there exists a bijection $f : X \to [n]$, then write $|X| = n$.

So finding bijections between sets of things and sets of natural numbers is the very **idea** of mathematically defining how to count things. It is the injective part that identifies each element of $X$ with a unique natural number, and the surjective part ensures that all the numbers in $\{1, 2, 3, \ldots, n\}$ are covered.

If the surjective part doesn't exist, then you have **inequalities** rather than equality. Thankfully, because of the way surjective functions are defined, you can 'shrink' the codomain of your function to the image to create a bijective function.

So if there exists an injective function $g : X \to [n]$, then this tells you that there is a bijection $g : X \to [m]$ where $m < n$. So not only do bijective functions tell you about the cardinality of sets, injective functions tell you which sets are bigger than others. The following are strengthenings of the above observations.

**i** Injections

(a) If there exists an injection $f : X \to Y$, then $|X| \leq |Y|$.

(b) If there exists an injection $g : Y \to X$, then $|Y| \leq |X|$.

(c) (**Cantor-Schröder-Bernstein theorem**) If there exists injections $f : X \to Y$ and $g : Y \to X$, then $|X| = |Y|$.

Notice here that none of the sets here were specifically stated to be **finite.** So this idea of comparing sets using injective and bijective functions extend to where the sets are **infinite** - and this will have even more surprising results in the next exploration...

## Version history

v1.0: initial version created 07/26 by tdhc, originally for presentation 3 of 4 for Sutton Trust Summer School 2026.